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European Congress on Computational Methods in Applied Sciences and Engineering

Contributions on the use of arbitrarily smooth generalized finite element approximation functions: application to crack modeling

Diego Amadeu F. Torres

Clovis S. de Barcellos

Paulo de Tarso R. Mendonça

UFSC

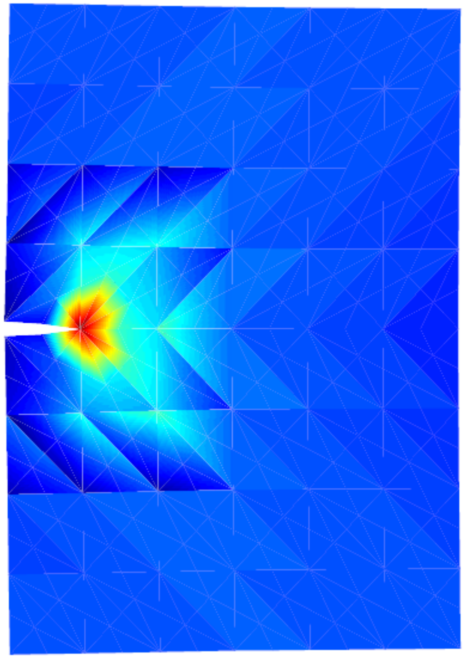
Group of Mechanical Analysis and Design

Department of Mechanical Engineering

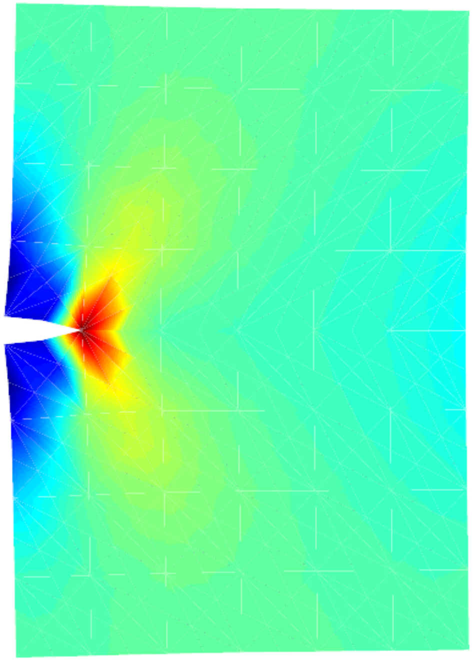
Federal University of Santa Catarina

Brazil

Motivation



?

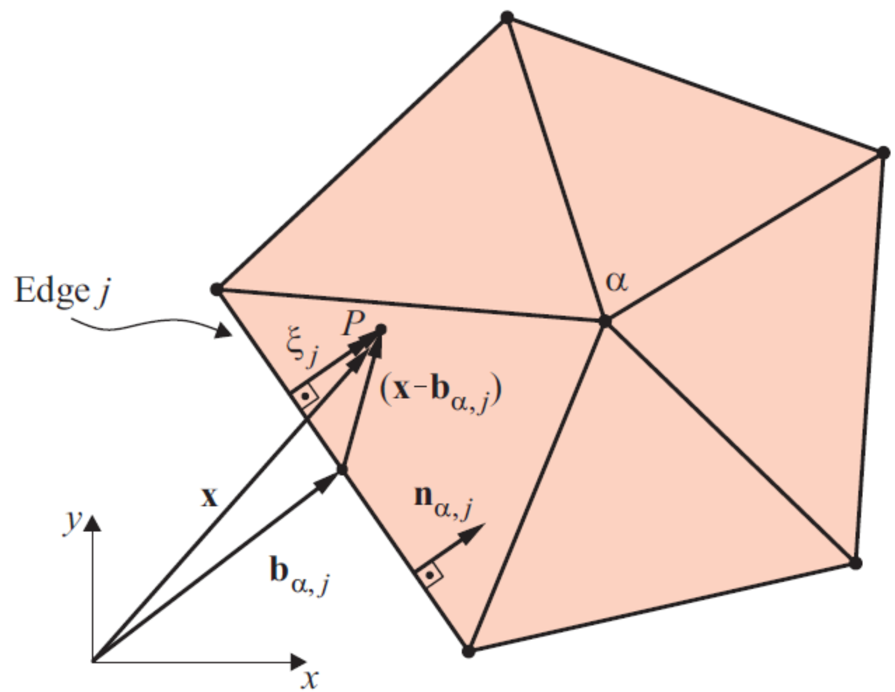


Why look forward continuous partition of unity?

Presentation topics

- Continuous partition of unity with GFEM
- Defining an approximation subspace
- Quality assessment through global measures
- Eshelbian mechanics
- Quality assessment through local measure
- Cloud-based residual error estimation
- Some improvements beyond...

C^∞ partition of unity – convex clouds

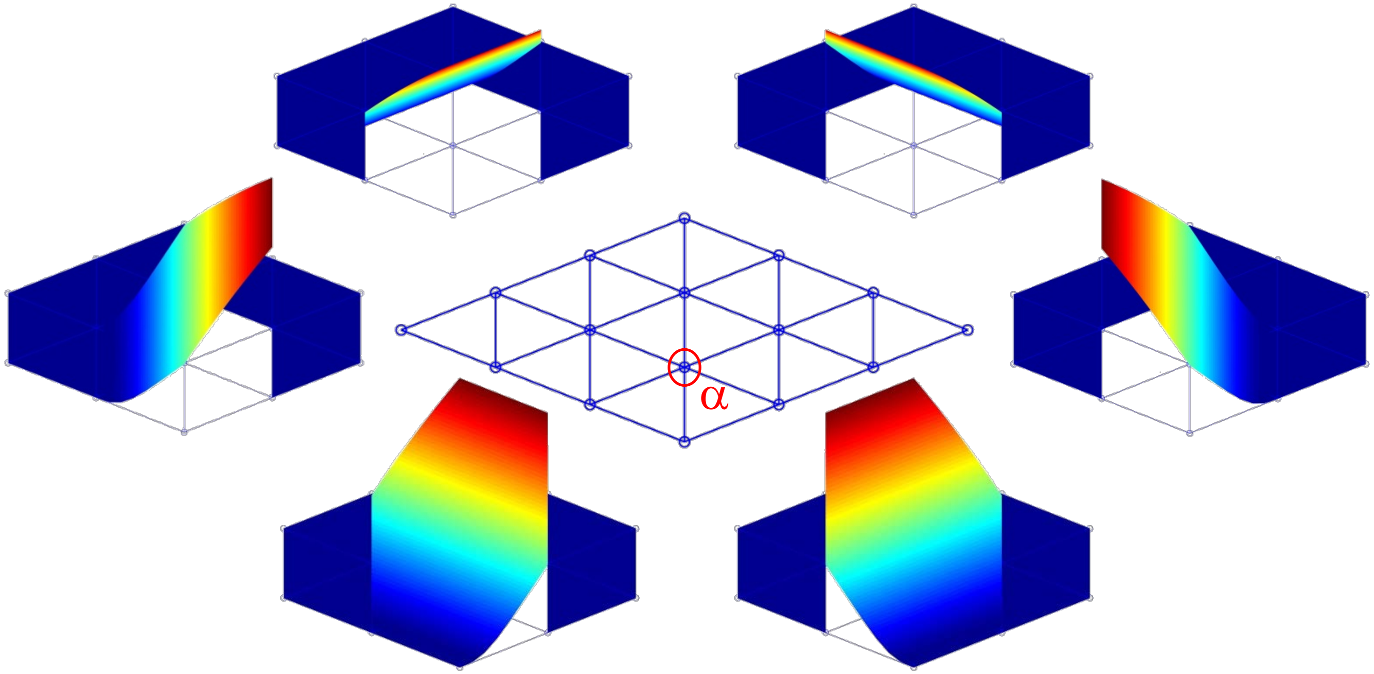


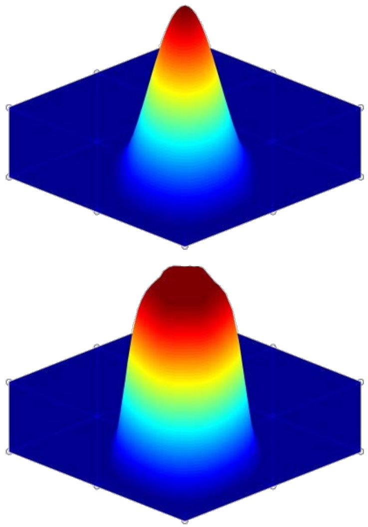
$$\varepsilon_{\alpha,j}\left[\xi_j(\mathbf{x})\right] = \widehat{\varepsilon}_{\alpha,j}(\mathbf{x}) := \begin{cases} e^{-\xi_j^{-\gamma}} & \text{if} \quad 0 < \xi_j \\ 0, & \text{otherwise} \end{cases}$$

$$\xi_j(\boldsymbol{x}) = \boldsymbol{n}_{\alpha,j} \cdot (\boldsymbol{x} - \boldsymbol{b}_{\alpha,j})$$

Edwards, **C^∞ finite element basis functions**, Report 45,
Institute for Computational Engineering and Sciences – The University of Texas at Austin, 2006

C^∞ partition of unity – convex clouds





$$\mathcal{W}_\alpha(\boldsymbol{x}) := \prod_{j=1}^{M_\alpha} \varepsilon_{\alpha,j}(\xi_j)$$

$M_\alpha = 6$

$$\varphi_\alpha(\boldsymbol{x}) = \frac{\mathcal{W}_\alpha(\boldsymbol{x})}{\sum_{\beta(\boldsymbol{x})} \mathcal{W}_\beta(\boldsymbol{x})},$$

$$\beta(\boldsymbol{x}) \in \{\gamma \mid \mathcal{W}_\gamma(\boldsymbol{x}) \neq 0\}$$

Continuous partition of unity with GFEM

Defining an approximation subspace

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Some improvements beyond

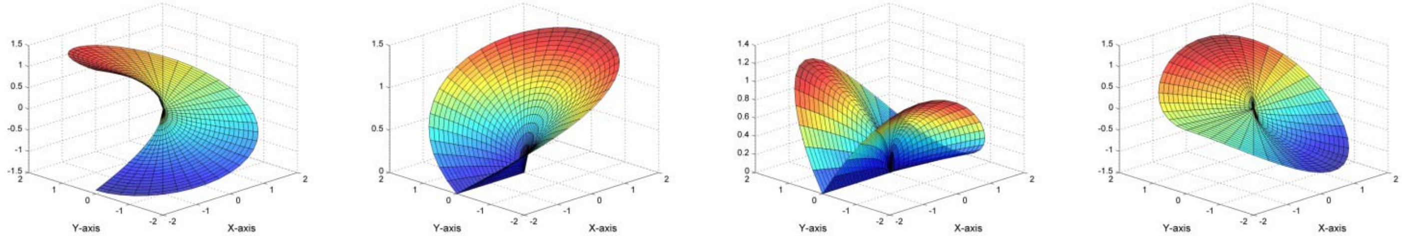
Galerkin aproximation

$$\boldsymbol{u}_p(\boldsymbol{x}) = \sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x}) \left\{ u_\alpha + \sum_{i=1}^{q_\alpha} \mathcal{L}_{\alpha i}(\boldsymbol{x}) b_{\alpha i} + \sum_{j=1}^{q_\alpha^s} \mathcal{L}_{\alpha j}^s b_{\alpha j}^s \right\}$$

if p=3 $\mathcal{L}_{\alpha 9}(x,y) = \left\{ \overline{x}, \overline{y}, \overline{x}^2, \overline{x}\,\overline{y}, \overline{y}^2, \overline{x}^3, \overline{x}^2\,\overline{y}, \overline{x}\,\overline{y}^2, \overline{y}^3 \right\}$

e.g. $\overline{x} := \dfrac{(x - x_\alpha)}{h_\alpha}$ for reducing mesh dependences

$q_\alpha^s = 4 \text{ or } 0$



$$\mathcal{L}_{\alpha 4}^s(r,\theta) = \left\{ \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right\}$$

Defining the degree of an approximation

Continuous partition of unity with GFEM

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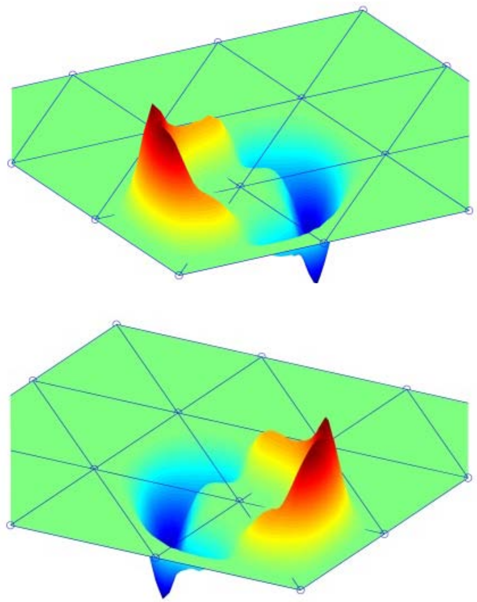
Some improvements beyond



b=p+1 for C^0 PoU (bilinear lagrangian shape function)

b=p for C^k PoU

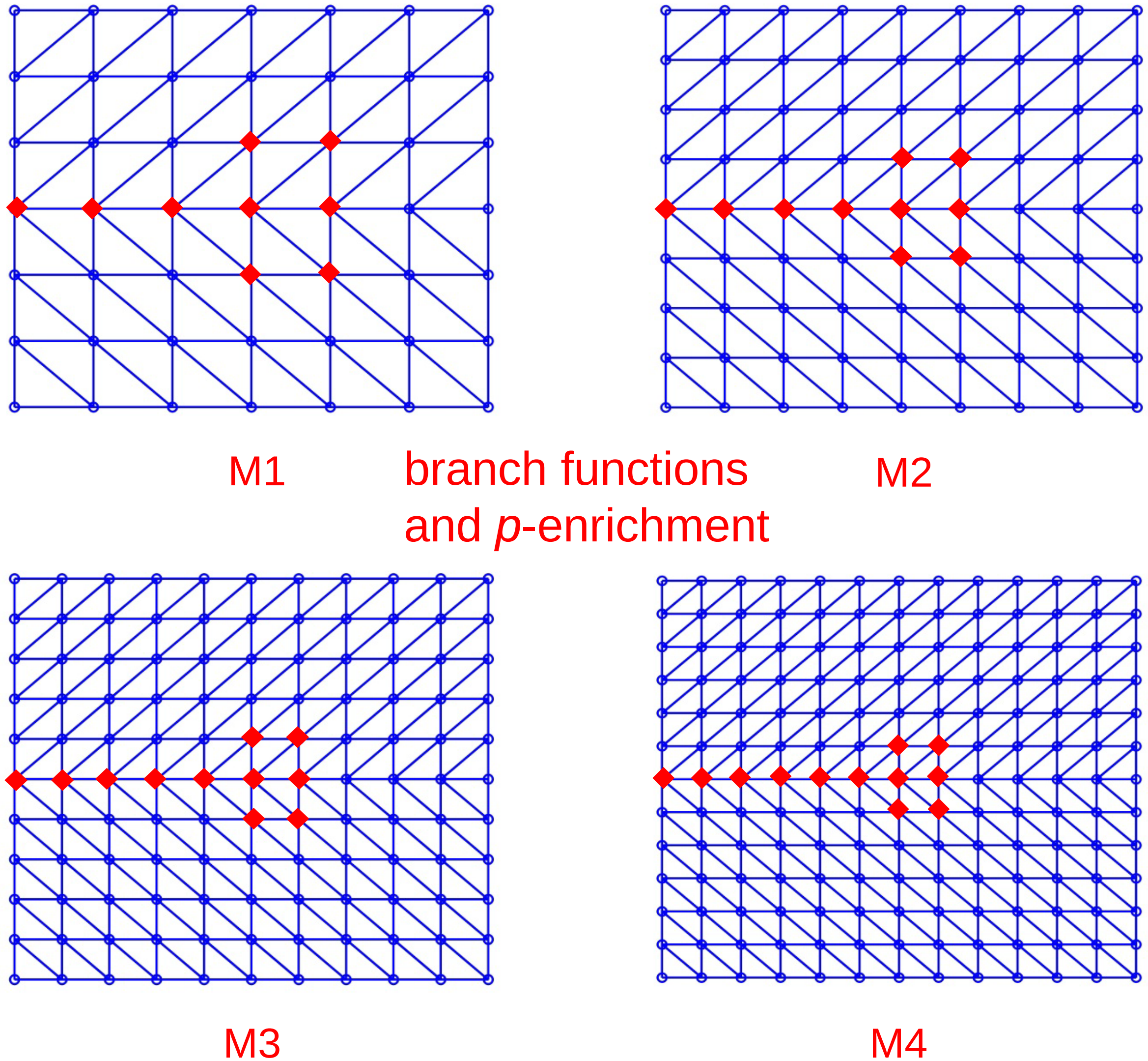
p = degree of polynomial enrichment

Mendonça, Barcellos, Torres, ***Robust Ck/C0 generalized FEM approximations for higher-order conformity requirements: application to Reddy's HSDT model for anisotropic laminated plates***. Accepted for publication in Composite Structures (2012)

Mendonça, Barcellos, Torres, ***Analysis of anisotropic Mindlin plate model by continuous and non-continuous GFEM.*** Finite Element in Analysis and Design, 47 (2011)

Barcellos, Mendonça, Duarte, ***A Ck continuous generalized finite element formulation applied to laminated Kirchhoff plate model.*** Computational Mechanics, 44 (2009)

Model problem – mode I crack opening



Continuous partition of unity with GFEM
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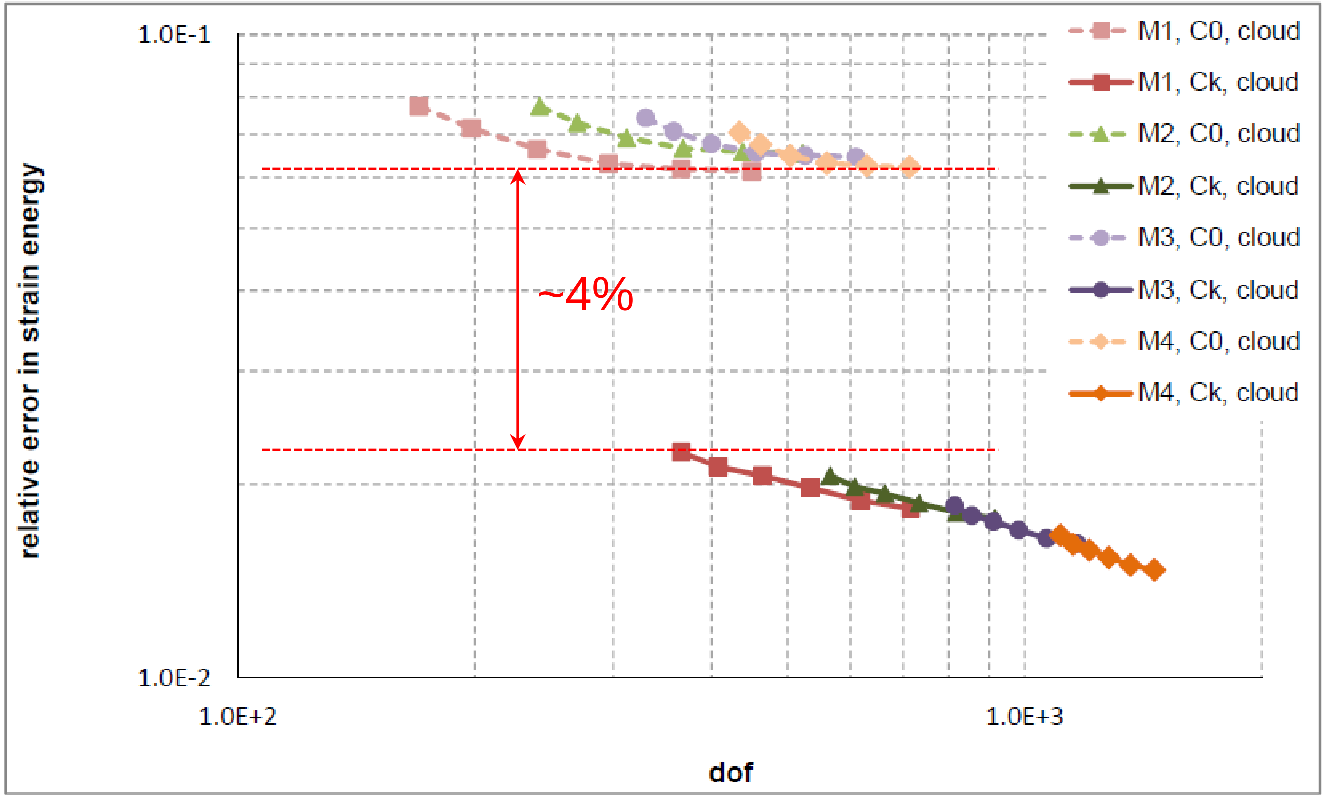
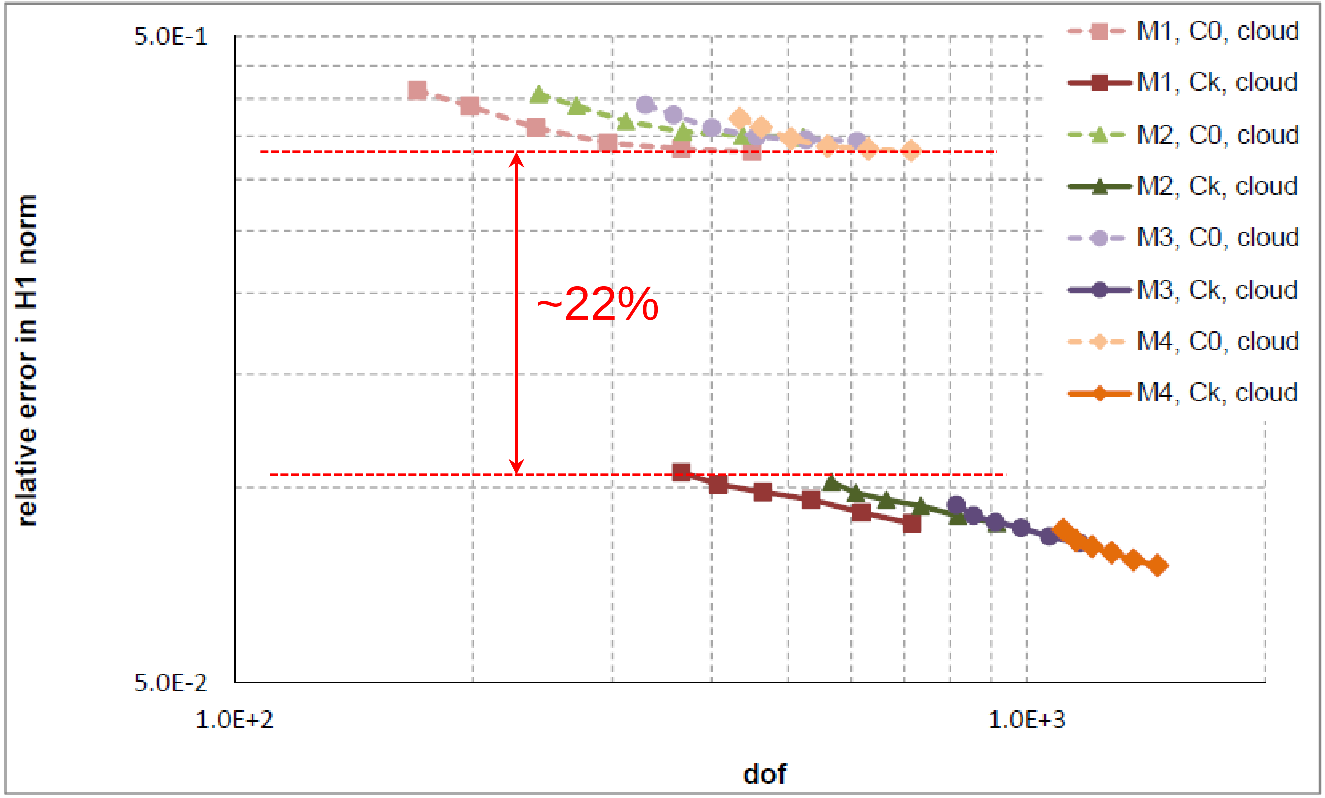
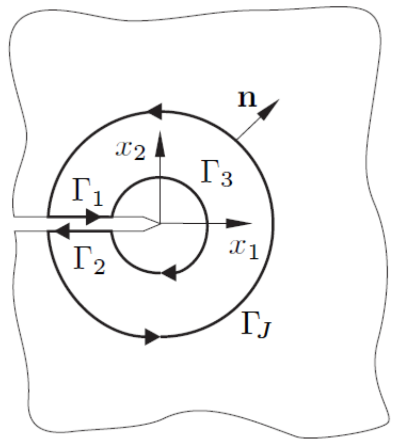
Global measure using strain energy

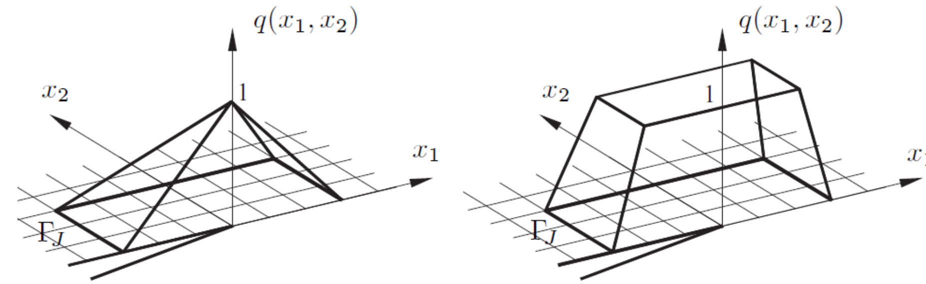


Global measure using H1-norm



Computation of the J-integral





$$J = \int_{\Gamma_J} \left(W dx_2 - \sigma_{ij} \mathbf{n}_j \frac{\partial u_i}{\partial x_1} \mathrm{d}\Gamma_J \right) = \int_{\Gamma_J} \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \mathbf{n}_j \mathrm{d}\Gamma_J$$

$$J = \int_{\Omega} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} \mathrm{d}\Omega$$

$$W(x_1, x_2) = \int_0^{\varepsilon} \sigma_{ij} \mathrm{d}\varepsilon_{ij}$$

$$G = J = \int_{\Gamma_J} \left(W \delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \mathbf{n}_j \mathrm{d}\Gamma_J = \frac{1}{E'} (K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2)$$

$$E' = \begin{cases} E \\ E / \left(1 - \upsilon^2\right) \end{cases}$$

Eshelbian mechanics

$$\mathfrak{W} = \int_0^{\varepsilon_{ij}} \sigma_{ij}\left(\bar{\varepsilon}_{ij}\right)\ d\bar{\varepsilon}_{ij} \qquad \mathfrak{W} = \mathfrak{W}\left(x_k, u_{j,i}\right)$$

$$\mathfrak{V} = -b_i u_i \qquad \mathfrak{V} = \mathfrak{V}\left(x_k, u_{i,j}\right)$$

$$\mathfrak{L} = -(\mathfrak{W} + \mathfrak{V}) = \mathfrak{L}\left(x_k, u_i, u_{i,j}\right)$$

$$\frac{\partial \mathfrak{L}}{\partial x_i} = -\frac{\partial\left(\mathfrak{W} + \mathfrak{V}\right)}{\partial x_i} = \varrho_i$$

inhomogeneity force

$$\Sigma_{ij} = (\mathfrak{W} + \mathfrak{V})\delta_{ij} - \sigma_{ik} u_{k,j}$$

Eshelby stresses

Kienzler, Herrmann, **Mechanics in material space with applications to defect and fracture mechanics.** Springer (2000)

Continuous partition of unity with GFEM

Defining an approximation subspace

Quality assessment through global measures

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Cloud-based residual error estimation

Some improvements beyond

$$\Sigma_{ji,j} = -\varrho_i$$

balance of material linear momentum

strong form

$$\int\int_\Omega \left(\boldsymbol{L}^T\boldsymbol{\Sigma} + \boldsymbol{\varrho}\right) \cdot \boldsymbol{v}\ l_z\ dx\ dy = 0$$

weak form

$$\boldsymbol{\varrho} = \{\varrho_x, \varrho_y\}^T$$

$$\boldsymbol{\Sigma} = \{\Sigma_x, \Sigma_y, \Sigma_{xy}, \Sigma_{yx}\}^T$$

$$\boldsymbol{v} = \{v_x, v_y\}^T$$

$$\boldsymbol{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

Mueller, Maugin, **On material forces and finite element discretizations.** Computational Mechanics, 29 (2000)

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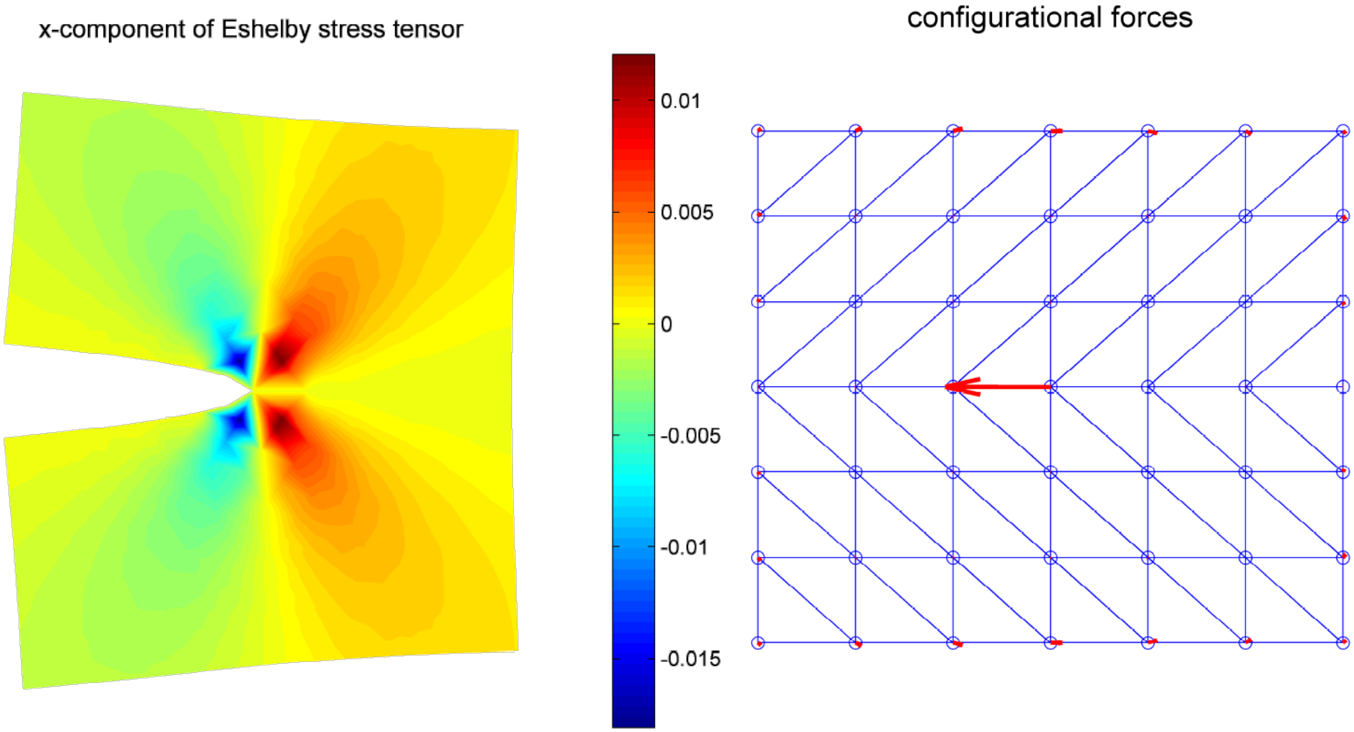
Local measure using Eshelby forces



Continuous partition of unity with GFEM

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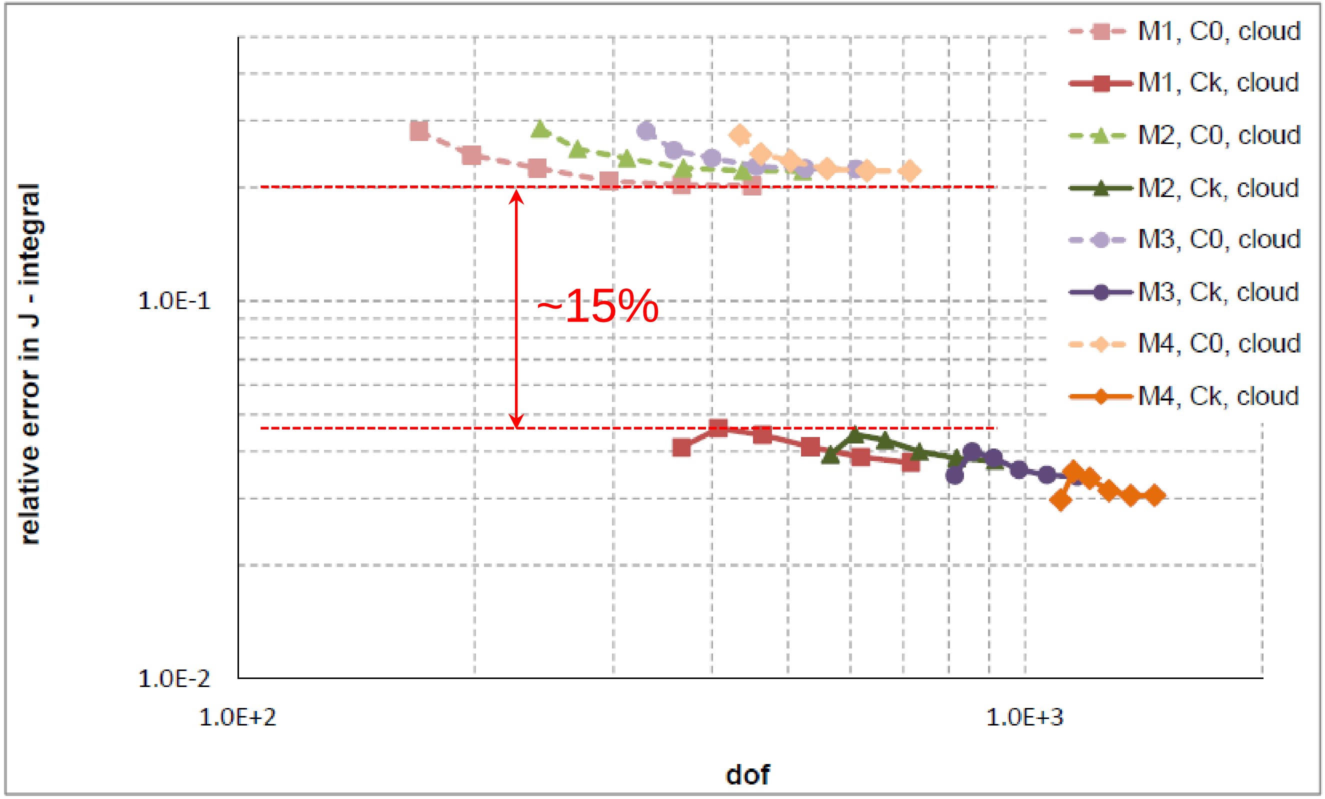
Local measure using configurational force



Stiffness matrix's conditioning number

Continuous partition of unity with GFEM

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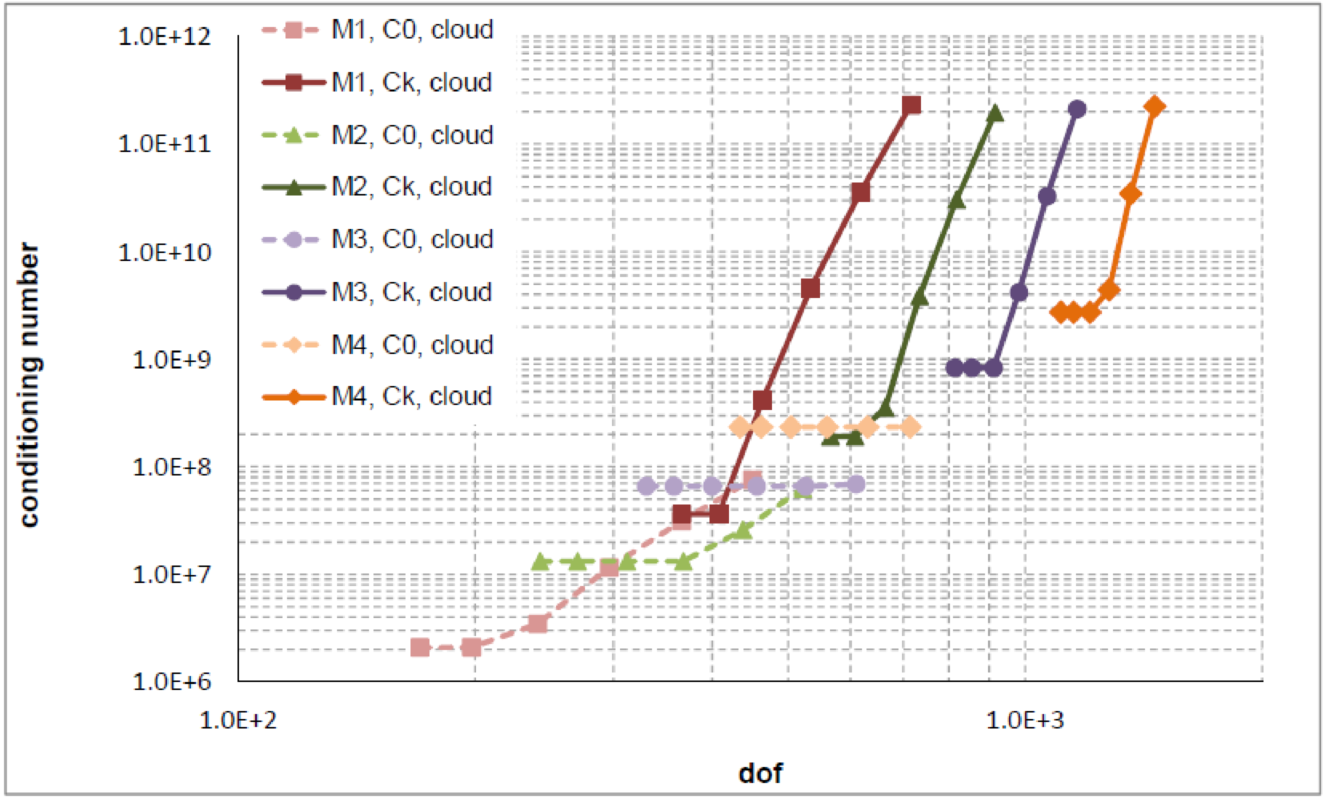

Continuous PoU on the whole domain!

Numerical integration's accuracy

Continuous partition of unity with GFEM

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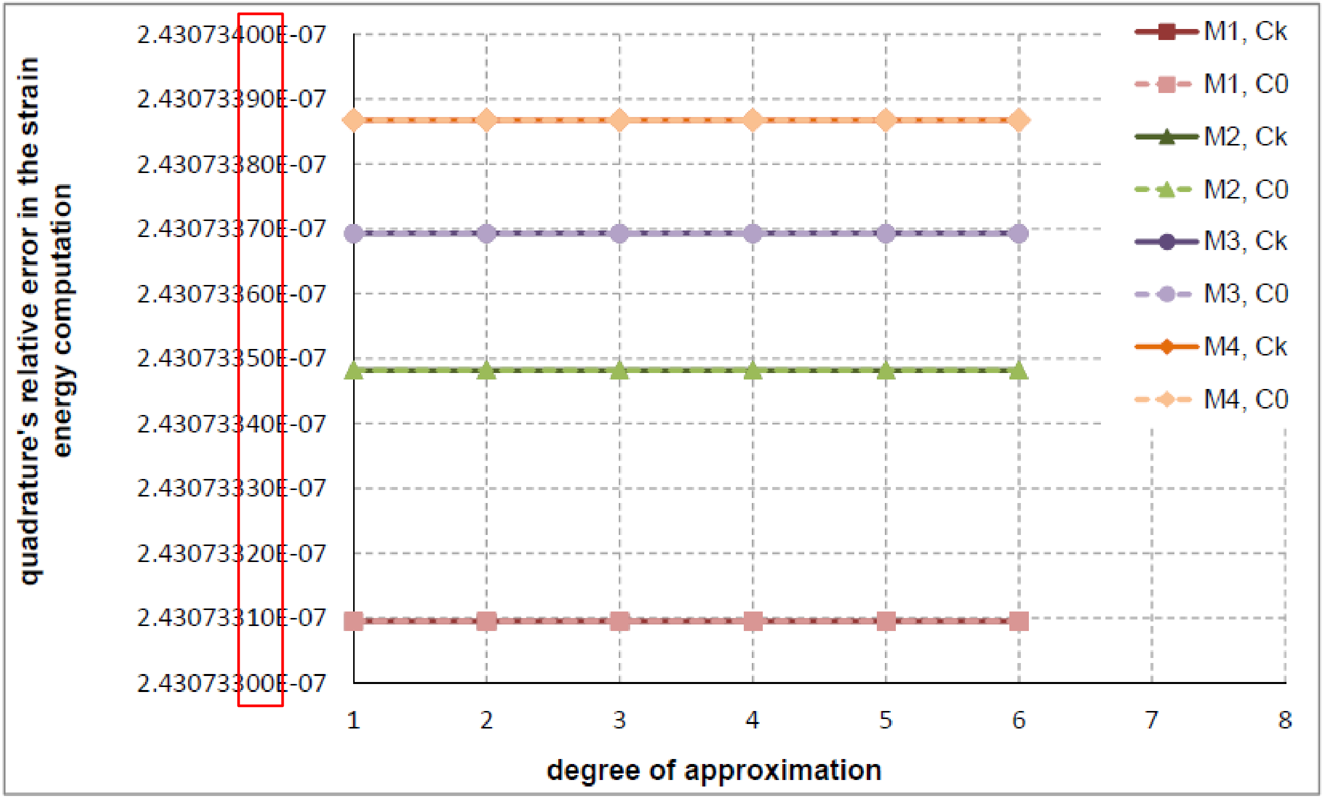

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Cloud-based implicit residual error estimator

$$\boldsymbol{e}_p = \boldsymbol{u} - \boldsymbol{u}_p$$

$$\mathscr{B}\left(\boldsymbol{e}_p, \boldsymbol{v}\right) = \mathscr{R}(\boldsymbol{v}) = \mathscr{R}\left(\boldsymbol{v}\sum_{\alpha=1}^{N}\varphi_\alpha\right) = \sum_{\alpha=1}^{N}\mathscr{R}\left(\varphi_\alpha \boldsymbol{v}\right)$$

Prudhomme, Nobile, Chamoin, Oden, **Analysis of a subdomain-based error estimator for finite element approximations of elliptic problems**. Numerical Methods for Partial Differential Equations, 20 (2004)

Strouboulis, Zhang, Wang and Babuska, **A posteriori error estimation for generalized finite element method**. Computer Methods in Applied Mechanics and Engineering, 195 (2006)

Parés, Díez and Huerta, ***Subdomain-based flux-free a posteriori error estimators***. Computer Methods in Applied Mechanics and Engineering, 195 (2006)

Barros, Barcellos and Duarte, ***Subdomain-based flux-free a posteriori estimator for generalized finite element method***. Proceedings of the thirth iberian-latin-american congress on computational methods in engineering – XXX CILAMCE (2009)

Barros, Barcellos, Duarte and Torres, ***Subdomain-based error techniques for GFEM approximations of problems with singular stress fields***. Submitted to Computational Mechanics (2012)

Strong-form and variational BVP

find $\boldsymbol{u} = \{u, v\}^T$

$$\boldsymbol{L}^T\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \ \text{ in } \Omega$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \text{ on } \Gamma_D$$

$$\boldsymbol{t}(\boldsymbol{u}) = \boldsymbol{\sigma}\boldsymbol{n} = \overline{\boldsymbol{t}} \text{ on } \Gamma_N$$

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{v}) = \mathscr{L}(\boldsymbol{v})$$

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{v}) := \int\int_\Omega \boldsymbol{\varepsilon}^T(\boldsymbol{v})\, \boldsymbol{\sigma}(\boldsymbol{u})\, l_z \, dx \, dy$$

$$\mathscr{L}(\boldsymbol{u}) := \int\int_\Omega \boldsymbol{v}^T\, \boldsymbol{b}\, l_z \, dx \, dy + \int_{\Gamma_N} \boldsymbol{v}^T\, \overline{\boldsymbol{t}}\, l_z \, ds$$

$$\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T$$

$$\boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$$

$$\boldsymbol{b} = \{b_x, b_y\}^T$$

$$\boldsymbol{t} = \{t_x, t_y\}^T$$

$$\boldsymbol{\varepsilon} = \boldsymbol{L}\boldsymbol{u}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

$$\boldsymbol{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

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Variational representation for the error

$$\boldsymbol{e}_p = \boldsymbol{u} - \boldsymbol{u}_p$$ error of an approximation of degree *p*

$$\mathscr{B}_{\omega_\alpha}^{\zeta_\alpha}\left(\boldsymbol{e}_p^{\omega_\alpha}, \boldsymbol{v}^{\omega_\alpha}\right) = \mathscr{R}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{\omega_\alpha}\right) \ \forall\, \boldsymbol{v}^{\omega^\alpha} \in \mathscr{V}(\omega_\alpha)$$

$$\mathscr{R}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{\omega_\alpha}\right) = \mathscr{L}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{\omega_\alpha}\right) - \mathscr{B}_{\omega_\alpha}\left(\boldsymbol{u}_p, \varphi_\alpha \boldsymbol{v}^{\omega_\alpha}\right)$$

$$\mathscr{B}(\bullet, \bullet) := \sum_{\alpha=1}^{N} \mathscr{B}_{\omega_\alpha}^{\zeta_\alpha}(\bullet, \bullet) \qquad \mathscr{V}_{brok} := \oplus_{\alpha=1}^{N} \mathscr{V}(\omega_\alpha)$$

$$\mathscr{B}_{\omega_\alpha}^{\zeta_\alpha}\left(\boldsymbol{e}_p^{\omega_\alpha}, \boldsymbol{v}_{\omega_\alpha}\right) = \int\!\!\int_{\omega_\alpha} \zeta_\alpha\, \varepsilon^T\left(\boldsymbol{v}^{\omega_\alpha}\right)\, \sigma\left(\boldsymbol{e}_p^{\omega_\alpha}\right)\, l_z\, dx\, dy$$

$$\zeta_\alpha = \varphi_\alpha \qquad \zeta_\alpha = 1$$

Prudhomme, Nobile, Chamoin, Oden, **Analysis of a subdomain-based error estimator for finite element approximations of elliptic problems**. Numerical Methods for Partial Differential Equations, 20 (2004)

Strouboulis, Zhang, Wang and Babuska, **A posteriori error estimation for generalized finite element method**. Computer Methods in Applied Mechanics and Engineering, 195 (2006) find $\tilde{\boldsymbol{e}}_p^{\omega_\alpha} \in \chi^0_{p+q}(\omega_\alpha)$

$$\chi^0_{p+q}(\omega_\alpha) = \left\{ \boldsymbol{v}^{0,\omega_\alpha}_{p+q} \in \chi_{p+q}(\omega_\alpha);\ \Pi_p\left(\boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) = 0;\ \boldsymbol{v}^{0,\omega_\alpha}_{p+q} = 0 \text{ on } \partial\omega_\alpha \cap \Gamma_D \right\}$$

$$\mathscr{B}^{\zeta_\alpha}_{\omega_\alpha}\left(\tilde{\boldsymbol{e}}_p^{\omega_\alpha}, \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) = \mathscr{R}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) \quad \forall\, \boldsymbol{v}^{0,\omega_\alpha}_{p+q} \in \chi^0_{p+q}(\omega_\alpha)$$

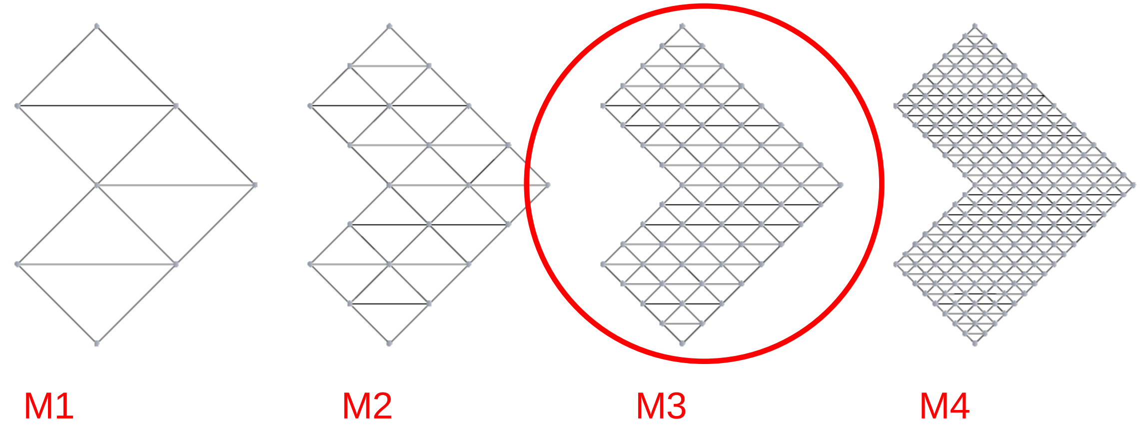
but

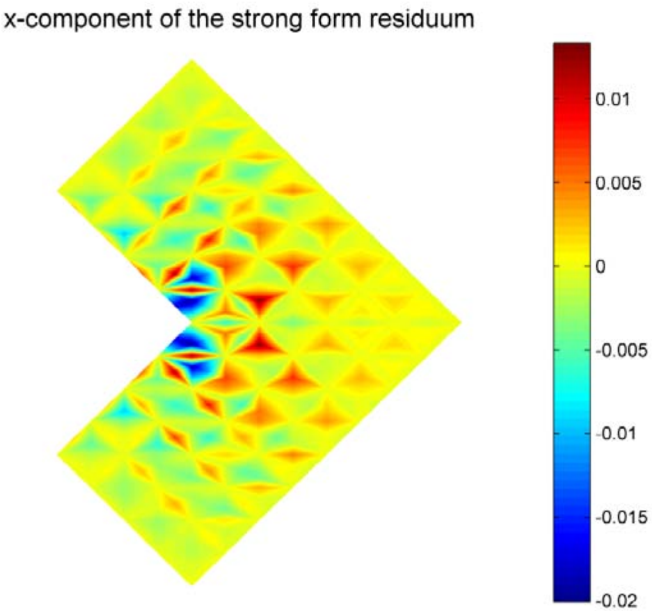
$$\mathscr{R}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) = \mathscr{L}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) - \mathscr{B}_{\omega_\alpha}\left(\boldsymbol{u}_p, \varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right)$$

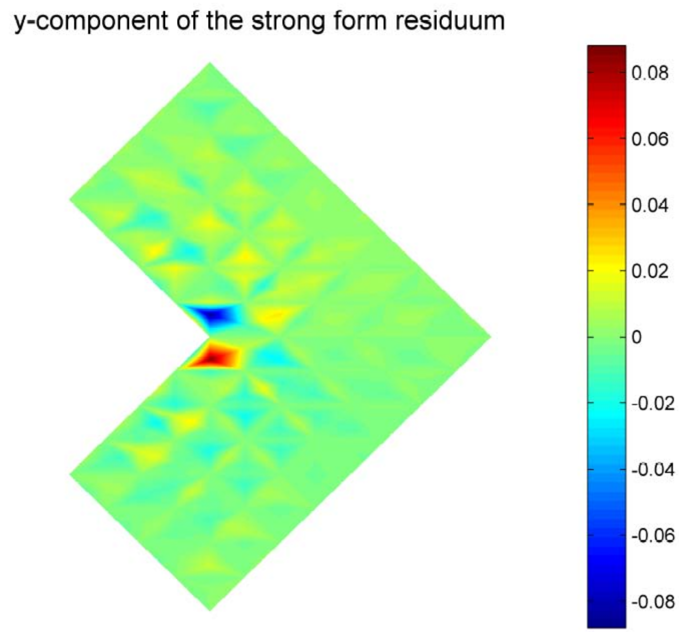
replaced with

$$\mathscr{R}_{\omega_\alpha}\left(\varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right) = \int\int_{\omega_\alpha} \left(\varphi_\alpha \boldsymbol{v}^{0,\omega_\alpha}_{p+q}\right)^T \boldsymbol{R}(\boldsymbol{u}_p)\, l_z\, dx\, dy$$

$$\boldsymbol{R}(\boldsymbol{u}_p) = \boldsymbol{L}^T \boldsymbol{\sigma}(\boldsymbol{u}_p) + \boldsymbol{b}$$

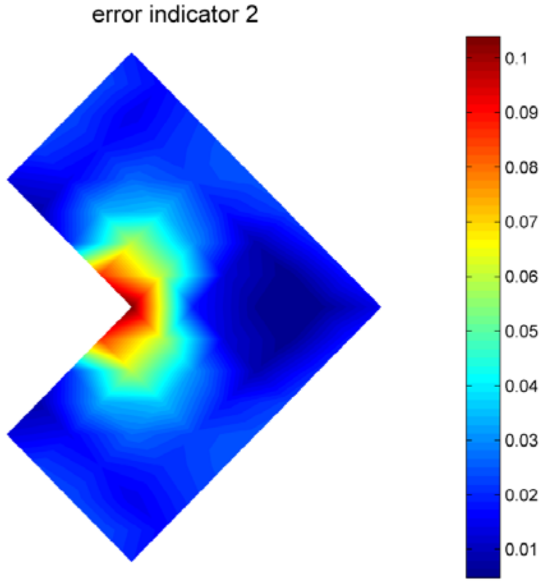


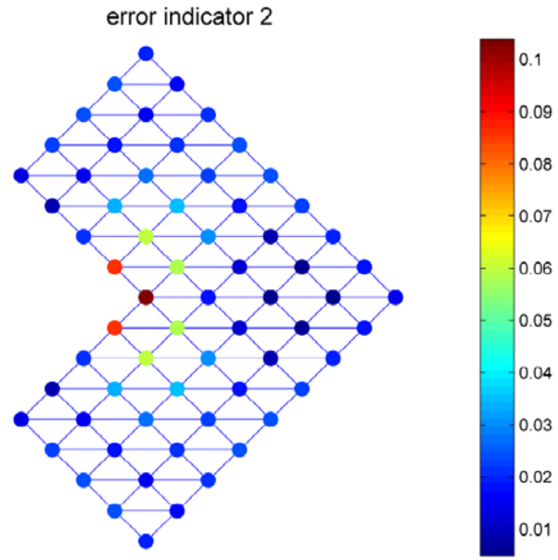


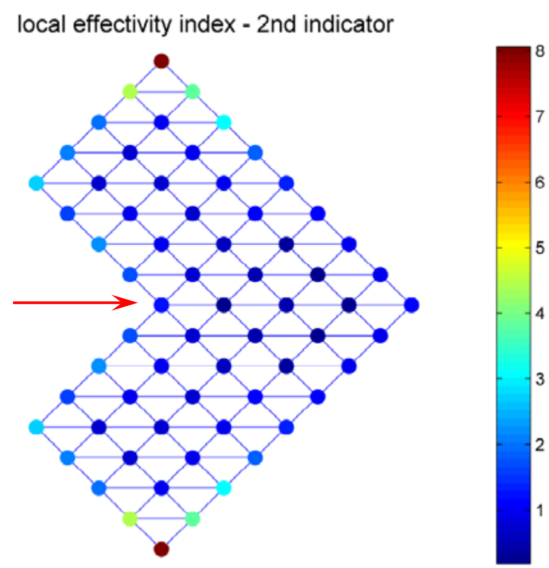


b=p=1

$$\boldsymbol{R}(\boldsymbol{u}_p) = \boldsymbol{L}^T \boldsymbol{\sigma}(\boldsymbol{u}_p) + \boldsymbol{b}$$







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$$\zeta_\alpha = 1$$

b=p=1

q=2

Strouboulis et al. (2006)

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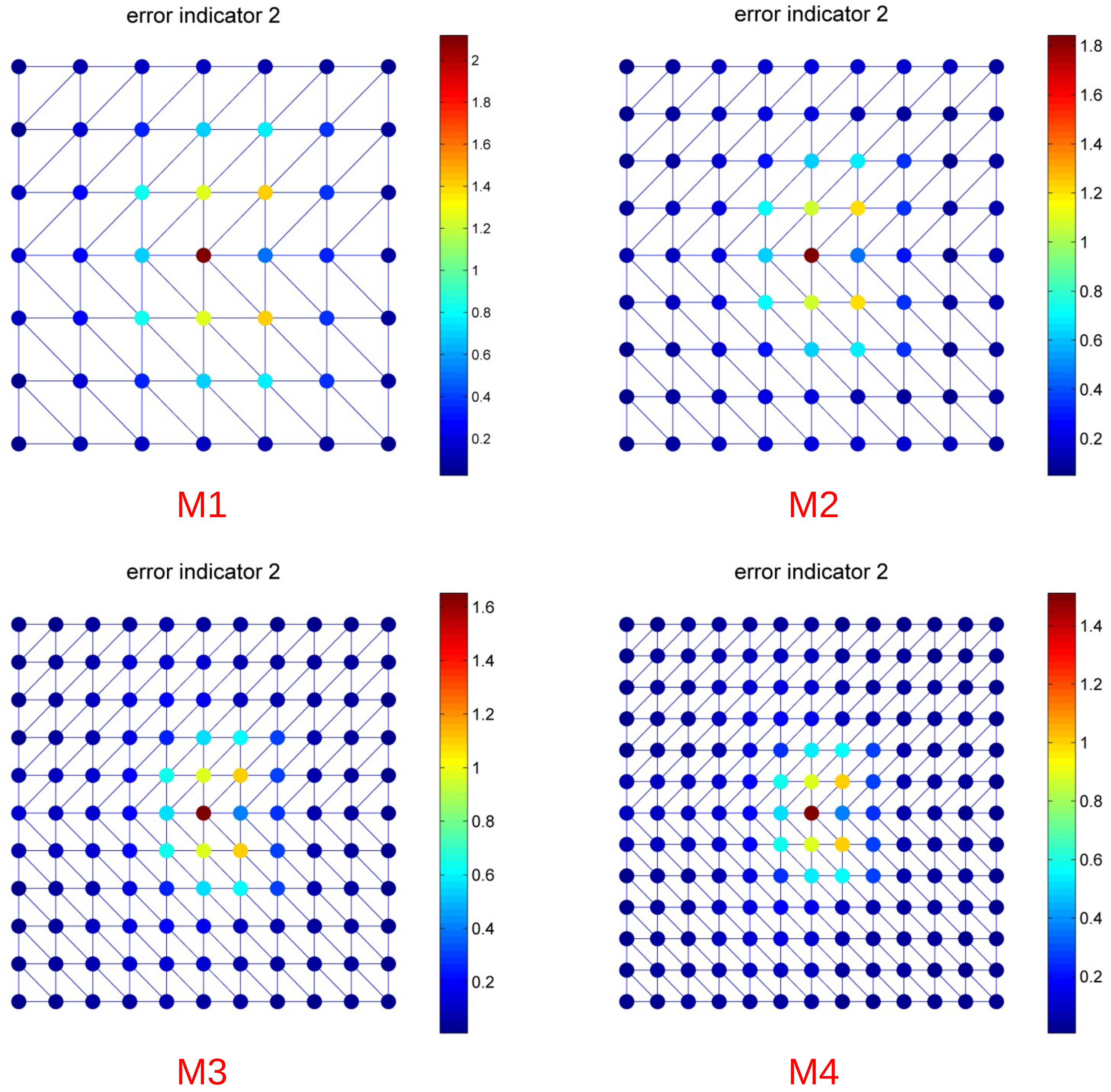
error indicator 2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
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M1
error indicator 2
1.8
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1.2
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0.8
0.6
0.4
0.2
M2
error indicator 2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
M3
error indicator 2
1.4
1.2
1
0.8
0.6
0.4
0.2
M4

Continuous partition of unity with GFEM

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Some improvements beyond

Implicity representation of geometry

R – fuctions theory

Encode complete logical information within the standard setting of real analysis

if α = 1

$$f_1 \wedge_\alpha f_2 \equiv \frac{1}{1+\alpha}\left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2}\right)$$ minimun

$$f_1 \vee_\alpha f_2 \equiv \frac{1}{1+\alpha}\left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2 - 2\alpha f_1 f_2}\right)$$ maximum

$$-1 \leq \alpha(f_1, f_2) \leq 1$$

Rvachev, **On the analytical description of some geometric objects**.
Reports of Ukrainian Academy of Sciences, 153 (1963)

Rvachev, Sheiko, Shapiro and Tsukanov, **Transfinite interpolation over implicitly defined sets**.
Computer Aided Design, 153 (2001)

Shapiro, **Semi-analytic geometry with R-functions**.
Acta Numerica (2007)

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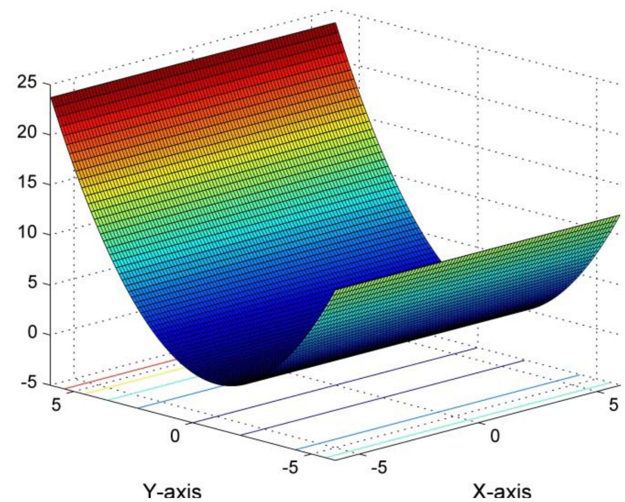
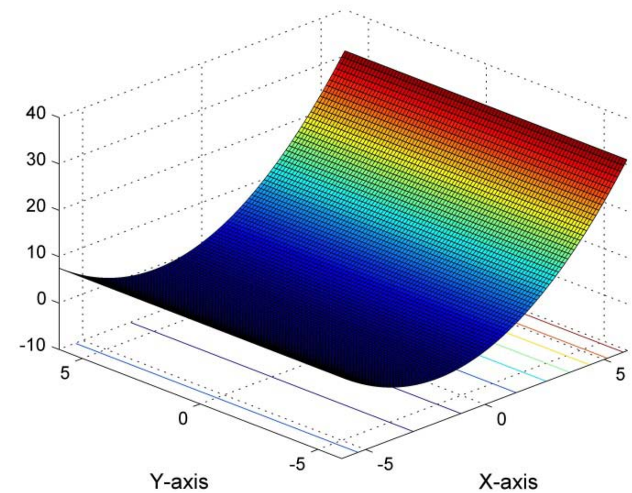
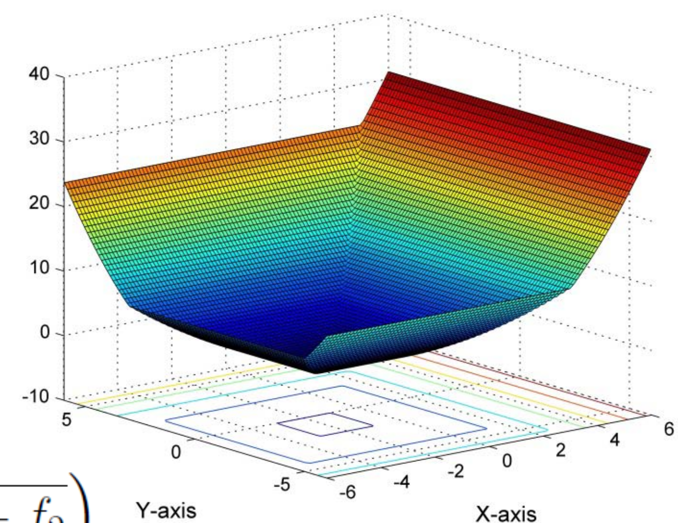




$$f_1(x,y) = -\frac{\left(\frac{h}{2}\right)^2 - (y-y_b)^2}{h}$$

$$f_2(x,y) = -\frac{\left(\frac{l}{2}\right)^2 - (x-x_b)^2}{l}$$

$(x_b, y_b) =$ center of the rectangle



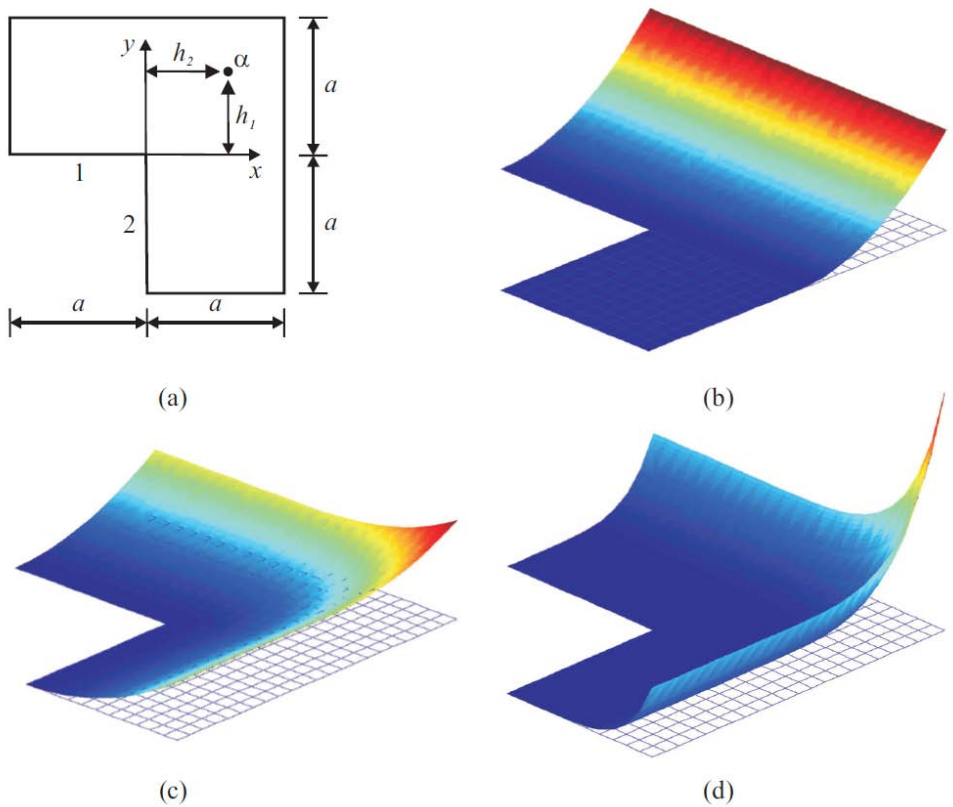
maximum

if $\alpha = 0$

$$f_1 \vee_0 f_2 = \left(f_1 + f_2 + \sqrt{f_1 + f_2}\right)$$

C^k partition of unity – non-convex clouds

$$\left(f_1 \vee_0^k f_2\right) := \left(f_1 + f_2 + \sqrt{f_1^2 + f_2^2}\right)\left(f_1^2 + f_2^2\right)^{\frac{k}{2}}$$



Duarte, Migliano, Quaresma, **Arbitrarily smooth generalized finite element approximations**. Computer Methods in Applied Mechanics and Engineering, 196 (2006)

Mendonça, Barcellos, Torres, ***Analysis of anisotropic Mindlin plate model by continuous and non-continuous GFEM.*** Finite Element in Analysis and Design, 47 (2011)

Coupling between mesh-based and meshfree PoU

$$\varphi_\alpha(\boldsymbol{x}) = \begin{cases} \dfrac{\mathcal{W}_\alpha^{fe}(\boldsymbol{x})}{\sum_{\beta\in\mathcal{I}_{fe}(\boldsymbol{x})}\mathcal{W}_\beta^{fe}(\boldsymbol{x}) + \sum_{\gamma\in\mathcal{I}_{mf}(\boldsymbol{x})}\mathcal{W}_\gamma^{mf}(\boldsymbol{x})}, \text{ if } \alpha\in\mathcal{I}_{fe} \\ \dfrac{\mathcal{W}_\alpha^{mf}(\boldsymbol{x})}{\sum_{\beta\in\mathcal{I}_{fe}(\boldsymbol{x})}\mathcal{W}_\beta^{fe}(\boldsymbol{x}) + \sum_{\gamma\in\mathcal{I}_{mf}(\boldsymbol{x})}\mathcal{W}_\gamma^{mf}(\boldsymbol{x})}, \text{ if } \alpha\in\mathcal{I}_{mf} \end{cases}$$

$$\mathcal{I}_{fe}(\boldsymbol{x}) = \{\beta\in\mathcal{I}_{fe} : \mathcal{W}_\beta(\boldsymbol{x})\neq 0\}$$
$$\mathcal{I}_{mf}(\boldsymbol{x}) = \{\beta\in\mathcal{I}_{mf} : \mathcal{W}_\beta(\boldsymbol{x})\neq 0\}$$

$$\mathfrak{I}_{mf,fe} = \mathfrak{I}_{fe}\cup\mathfrak{I}_{mf}$$

$$\{\varphi_\alpha\}_{\alpha\in\mathcal{I}_{mf,fe}} \qquad \sum_{\alpha\in\mathcal{I}_{mf,fe}}\varphi_\alpha(\boldsymbol{x}) = 1 \quad \forall\,\boldsymbol{x}\in\Omega$$

Duarte, Migliano, Baker, **A technique to combine meshfree- and finite element-based partition of unity approximations**, Research Report, Department of Civil and Environmental Engineering – UIUC, 2005

UFSC

Continuous partition of unity with GFEM

Defining an approximation subspace

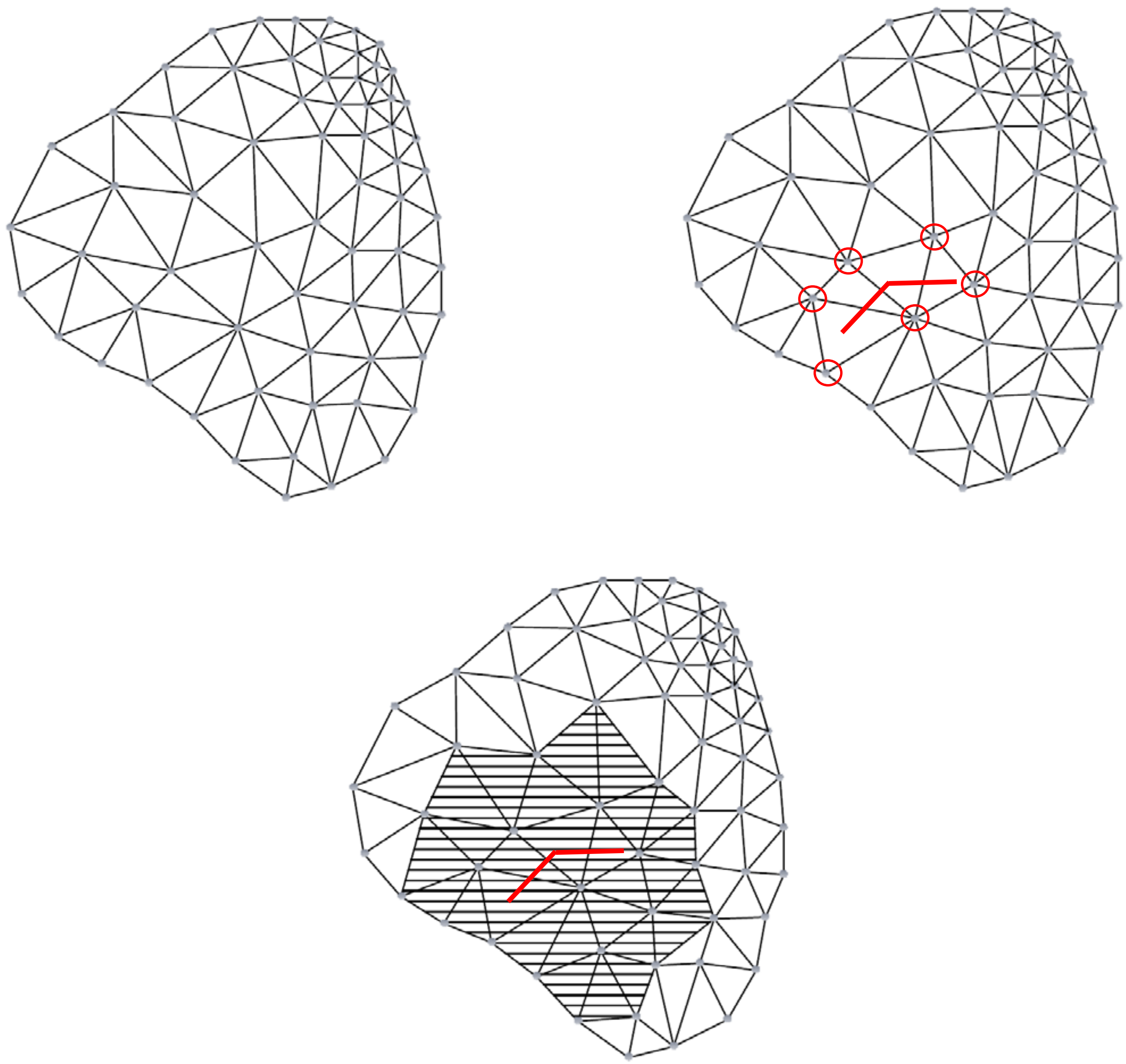
Quality assessment through global measures

Eshelbian mechanics

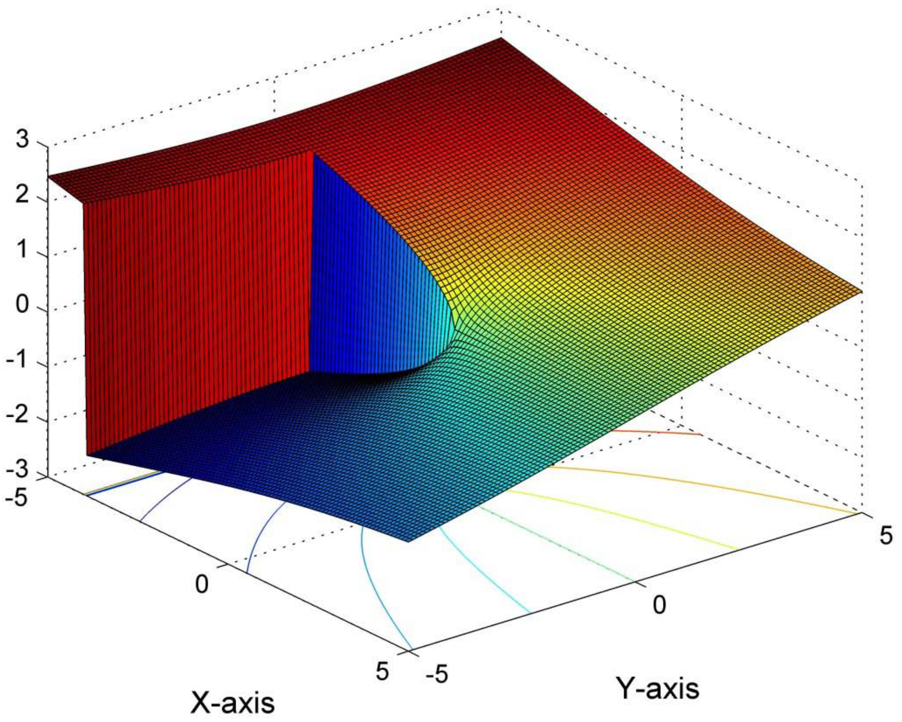
Quality assessment through local measure

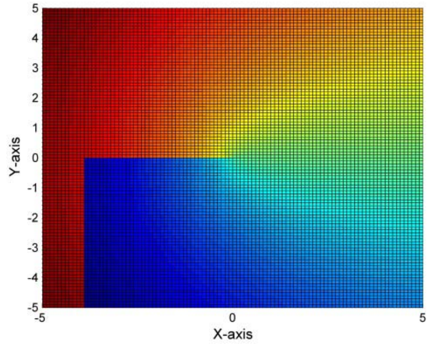
Cloud-based residual error estimation

Some improvements beyond



Enrichments building with R-functions





Concluding remarks

- Continuous stress fields around singularity provide better severity crack parameters

- Polynomial enrichments together branch functions may adaptively improve the stress fields

- Continuity may conduct to better computation of nodal Eshelby forces

- The inner product of the residuum field with higher-order functions seens to be effective for estimating the error

Acknowledgements

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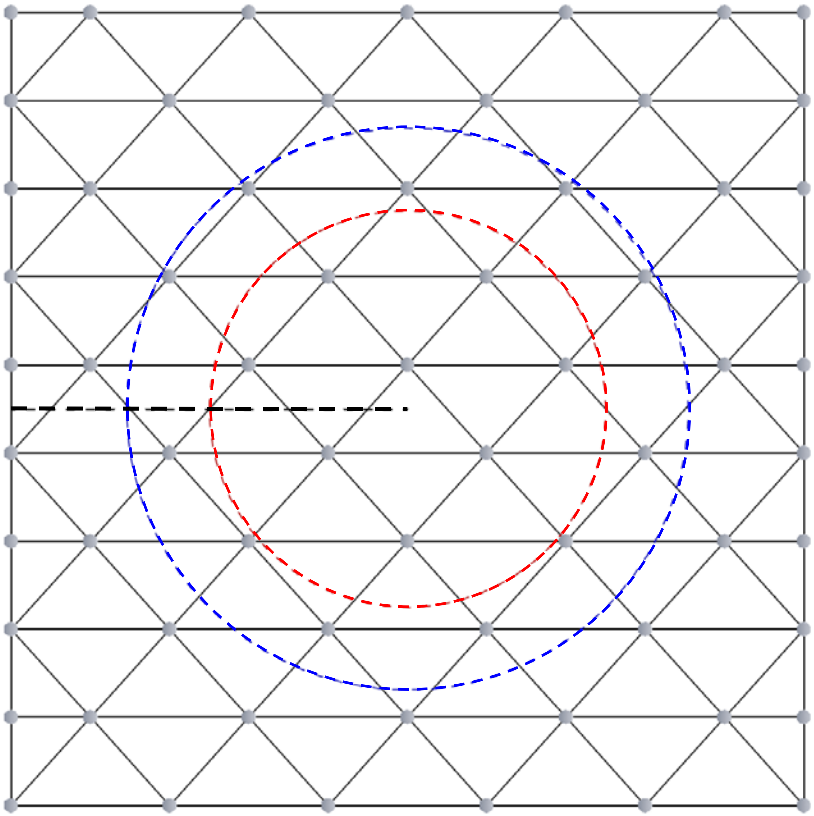
Thank you!

diego.amadeu@gmail.com

Some possible questions!

Some issues of concern in crack modeling

- Integration of singular functions
- Accuracy in computation of crack parameters
- Flexibility
- Rate of convergence
- Way to performe singular enrichment
- Blending
- Enrichments merging
- Conditioning of the stiffness matrix
- etc.



$$\mathscr{B}\left(\boldsymbol{e}_p, \boldsymbol{v}\right) = \mathscr{R}(\boldsymbol{v}) = \mathscr{R}\left(\boldsymbol{v}\sum_{\alpha=1}^{N}\varphi_\alpha\right) = \sum_{\alpha=1}^{N}\mathscr{R}\left(\varphi_\alpha \boldsymbol{v}\right)$$

$$\frac{\partial}{\partial \mathbf{n}}\left(\mathbf{e}_p^{\omega_j}\right) = 0 \quad \text{on } \partial\omega_j \setminus \left(\partial\omega_j \cap \Gamma_N\right)$$

$$\frac{\partial}{\partial \mathbf{n}}\left(\mathbf{e}_p^{\omega_j}\right) = \mathcal{N}_j\left(\mathbf{t} - \boldsymbol{\sigma}\left(\mathbf{u}_p\right)\mathbf{n}\right) \quad \text{on } \partial\omega_j \cap \Gamma_N$$

$$\left[\!\left[\frac{\partial}{\partial \mathbf{n}}\left(\mathbf{e}_p^{\omega_j}\right)\right]\!\right]_\gamma = \mathcal{N}_j\left[\!\left[\frac{\partial}{\partial \mathbf{n}}\left(\mathbf{u}_p\right)\right]\!\right]_\gamma$$

Strouboulis, Zhang, Wang, Babuska, **A posteriori error estimation for generalized finite element method***.* Computer Methods in Applied Mechanics and Engineering, 195 (2006)

Self-equilibrated enrichments

$$\delta W_{int} = \int_{\Omega^E} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \; d\Omega \qquad \delta \boldsymbol{\varepsilon} = \tfrac{1}{2} \nabla \delta \mathbf{u} + \tfrac{1}{2} \left(\nabla \delta \mathbf{u} \right)^t$$

$$\delta W_{int} = \frac{1}{2} \int_{\Omega^E} \boldsymbol{\sigma} : \nabla \delta \mathbf{u} \; d\Omega + \frac{1}{2} \int_{\Omega^E} \boldsymbol{\sigma} : \left(\nabla \delta \mathbf{u} \right)^t \; d\Omega.$$

divergence theorem

$$\delta W_{int} = - \int_{\Omega^E} \underbrace{\nabla \cdot \boldsymbol{\sigma}}_{=\mathbf{0}} \cdot \delta \mathbf{u} \; d\Omega + \int_{\partial \Omega^E} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \delta \mathbf{u} \; d\partial\Omega$$

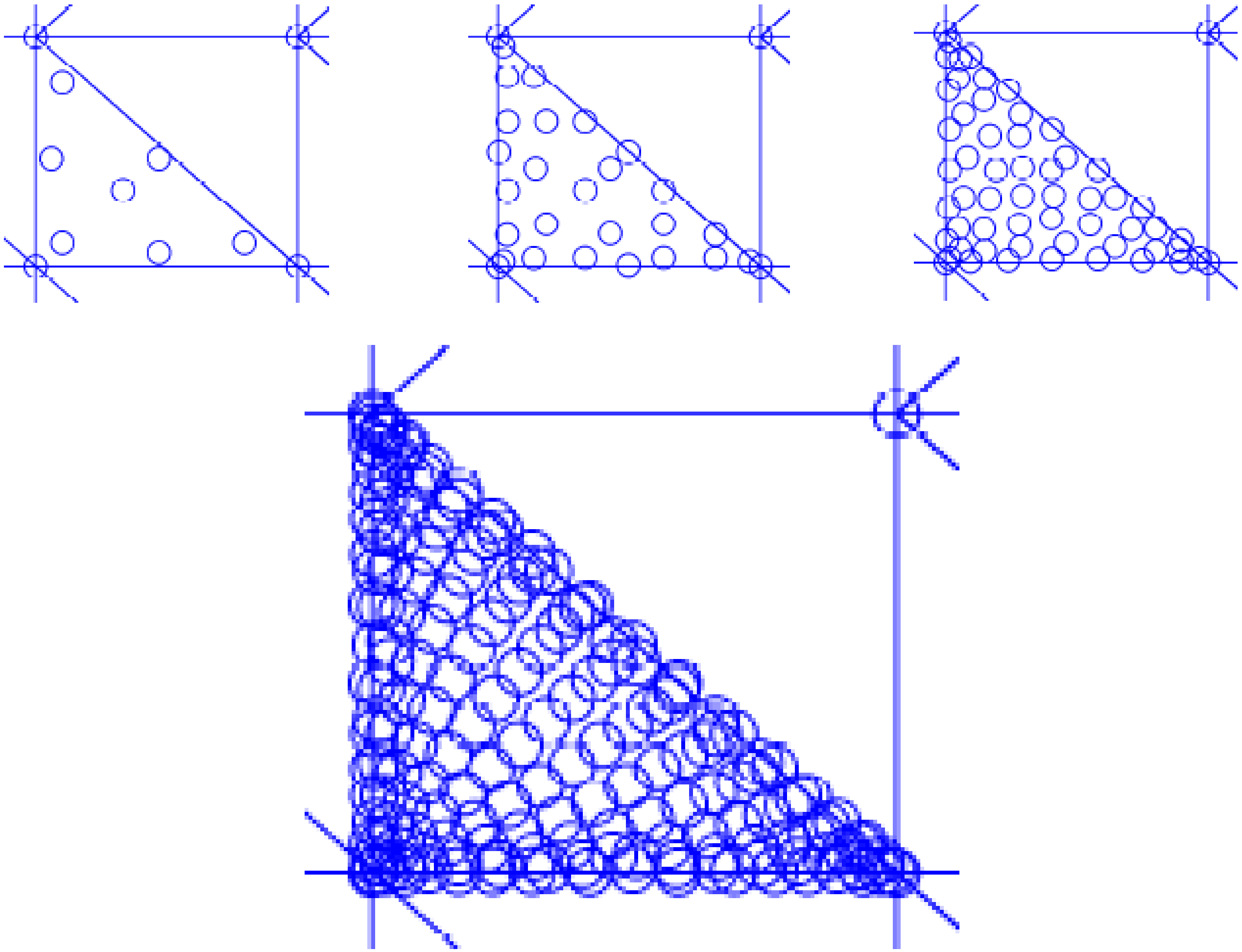
$$\delta W_{int} \quad = \quad \delta \mathbf{U}^t \left[\int_{\partial \Omega^E} \left[\mathbf{N}^E \right]^t [\mathbf{n}] \, [\mathbf{C}] \, \mathbf{B}^E \; d\partial\Omega \right] \mathbf{U}$$

E stands for “equilibrated”

$$\mathbf{K}^{EE} \quad = \quad \int_{\partial \Omega^E} \left[\mathbf{N}^E \right]^t [\mathbf{n}] \, [\mathbf{C}] \, \mathbf{B}^E \; d\partial\Omega$$

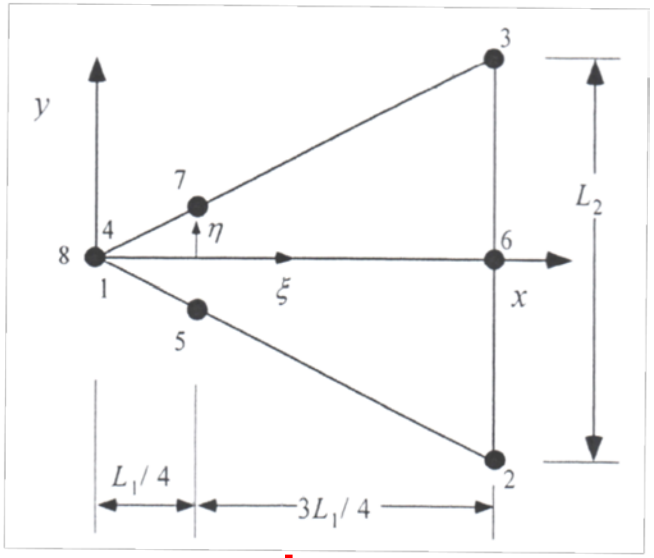
Ventura, Gracie, Belytschko, **Fast integration and weight function blending in the extended finite element method**. International Journal for Numerical Methods in Engineering, 77 (2009)

Numerical integration of regular functions

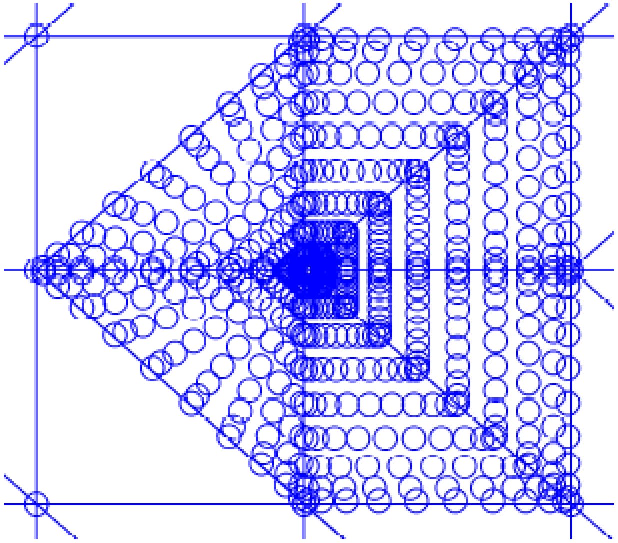


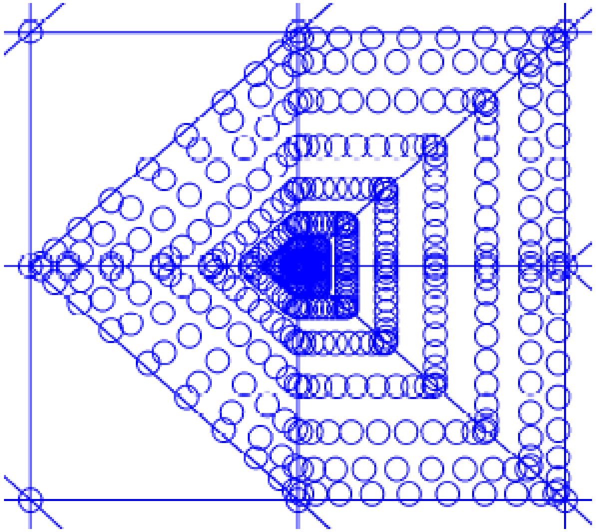
Wandzura and Xiao, ***Symmetric quadrature rules on a triangle.*** Computer and Mathematics with Applications, 45 (2003)

Numerical integration of singular functions



Quasi-polar mapping + Quarter-point mapping





225 points

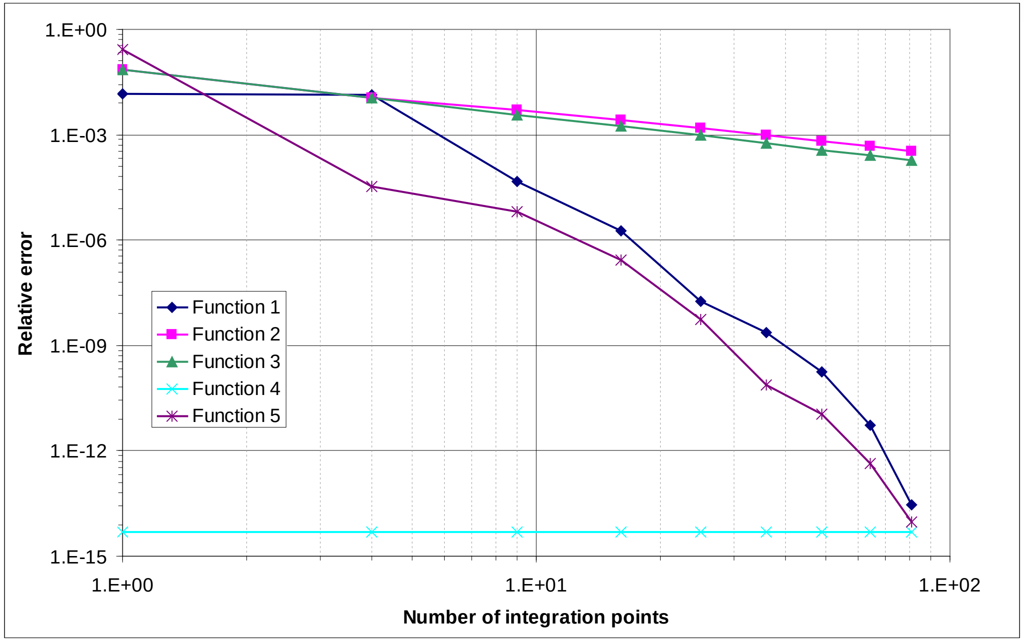
$$\text{function } 1 = \frac{1}{r}$$

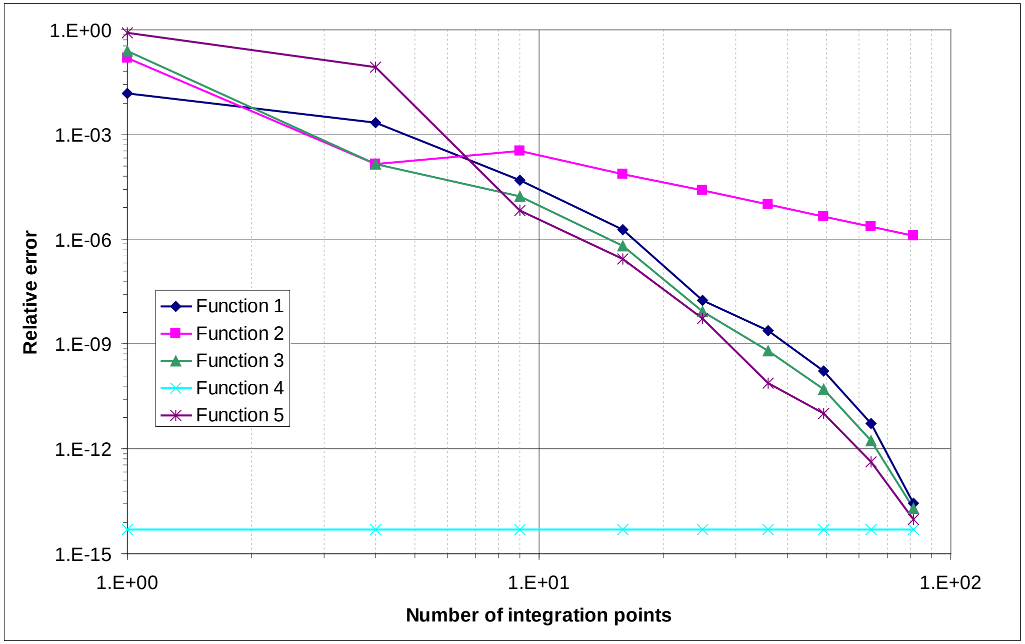
$$\text{function } 2 = \frac{1}{r^{\frac{2}{3}}}$$

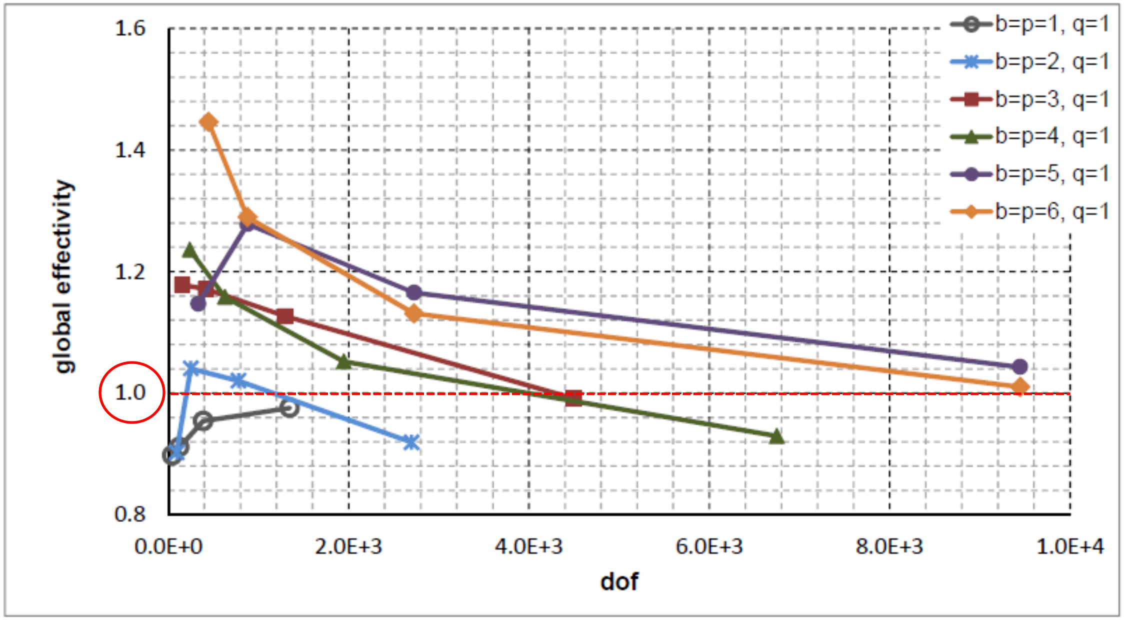
$$\text{function } 3 = \frac{1}{\sqrt{r}}$$

$$\text{function } 4 = 1$$

$$\text{function } 5 = r$$



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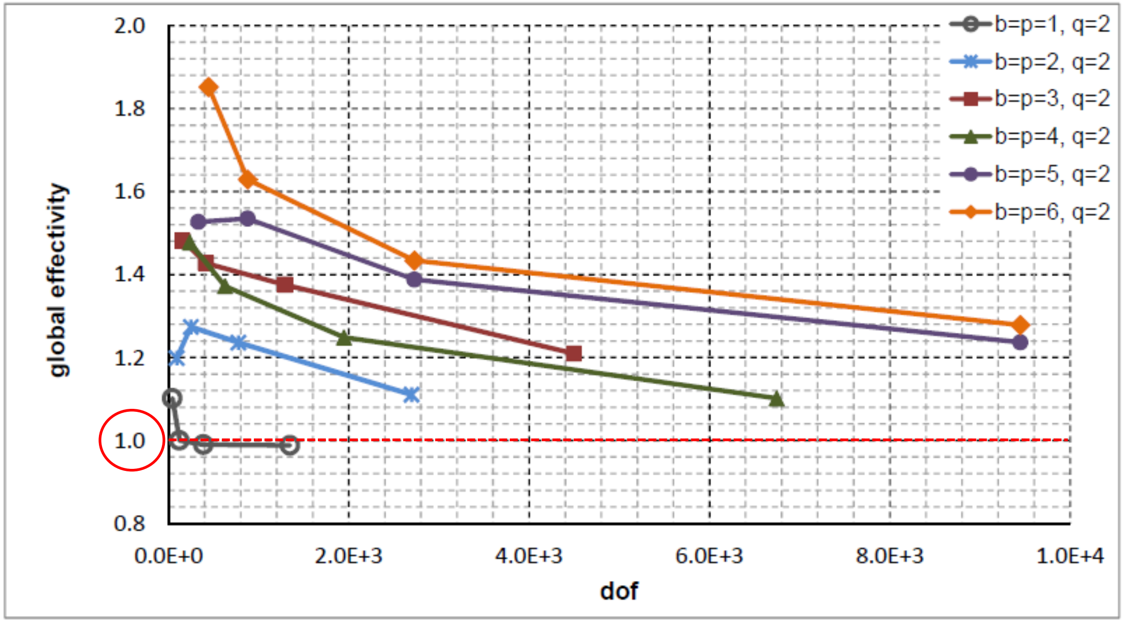
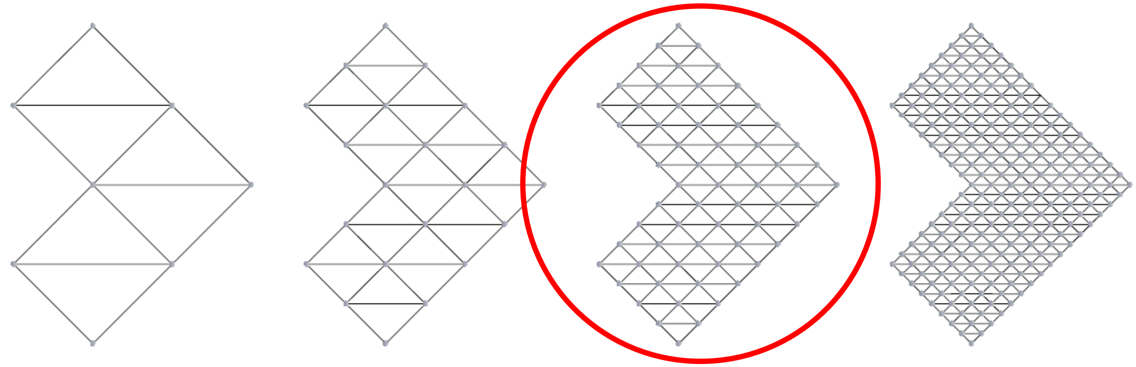
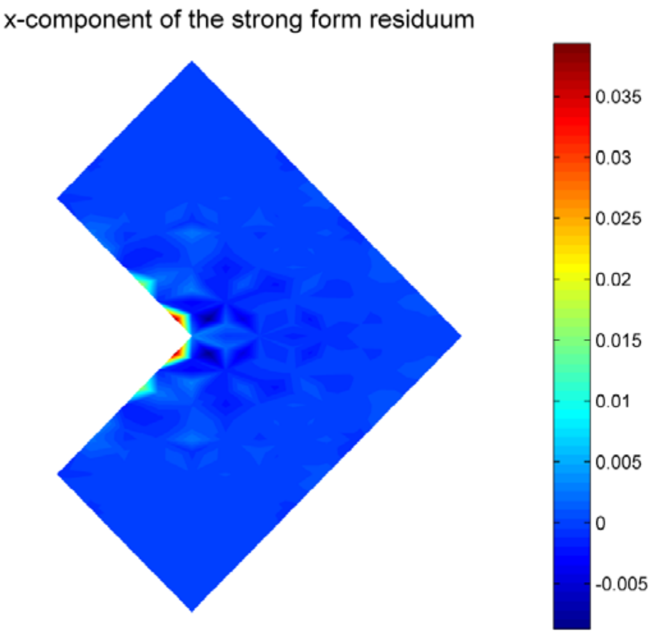
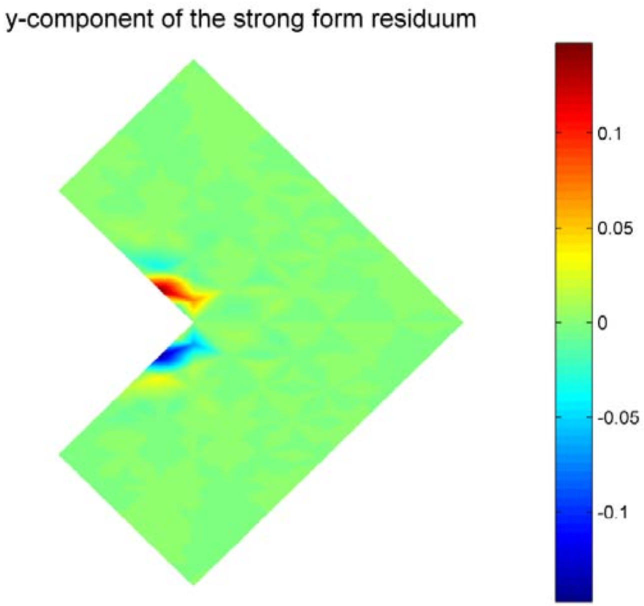
b=p=2, q=1
b=p=3, q=1
b=p=4, q=1
b=p=5, q=1
b=p=6, q=1
global effectivity
1.6
1.4
1.2
1.0
0.8
0.0E+0
2.0E+3
4.0E+3
6.0E+3
8.0E+3
1.0E+4
dof


b=p=1, q=2
b=p=2, q=2
b=p=3, q=2
b=p=4, q=2
b=p=5, q=2
b=p=6, q=2
global effectivity
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.0E+0
2.0E+3
4.0E+3
6.0E+3
8.0E+3
1.0E+4
dof



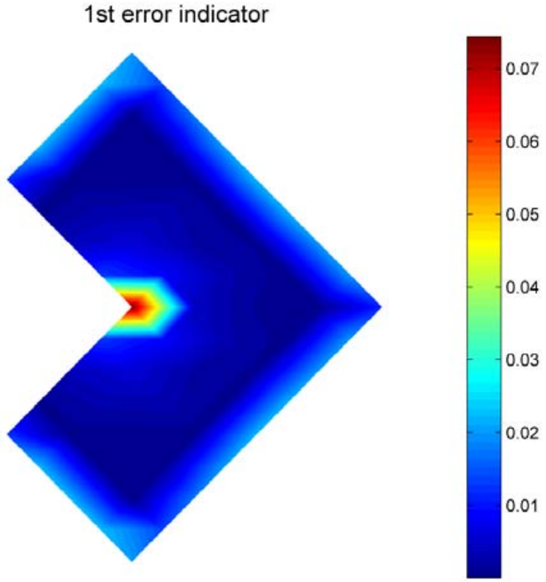
mesh 1 mesh 2 mesh 3 mesh 4

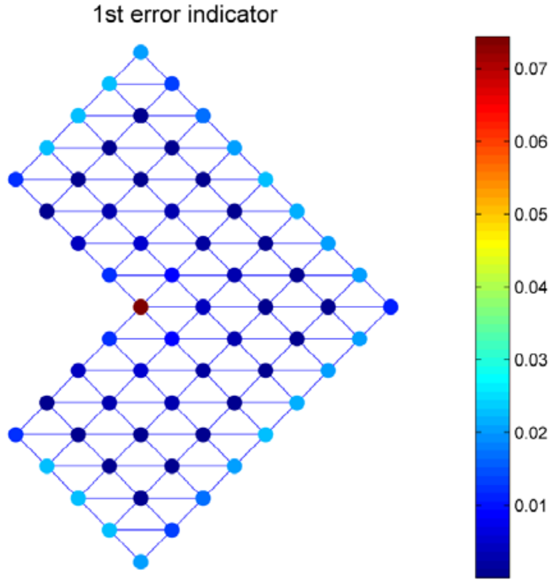




b=p=6

$$\boldsymbol{R}(\boldsymbol{u}_p) = \boldsymbol{L}^T \boldsymbol{\sigma}(\boldsymbol{u}_p) + \boldsymbol{b}$$


1st error indicator
0.07
0.06
0.05
0.04
0.03
0.02
0.01


1st error indicator
0.07
0.06
0.05
0.04
0.03
0.02
0.01

b=p=6

q=2

Comentar que a abordagem apresentada aqui é diferente do smoothed finite element method de Xuan, Lassila, Rozza e Quarteroni 2010 e verificar se este artigo usa o mesmo procedimento do apresentado em Bordas et al. 2011.

Xuan, Lassila, Rozza and Quarteroni, **On computing upper and lower bounds on the outputs of linear elasticity problems approximated by the smoothed finite element method**. International Journal for Numerical Methods in Engineering, 83 (2010)

Bordas, Natarajan, Kerfriden, Augarde, Mahapatra, Rabczuk and Dal Pont, **On the performance of strain smoothing for quadratic and enriched finite element approximations (XFEM / GFEM / PUFEM)**. International Journal for Numerical Methods in Engineering, 86 (2011)