Laminated composite plate analysis by GFEM using approximation functions with arbitrary inter-element continuity

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Broad objectives

- To develop a family of FE shape functions
  - with arbitrary inter-element continuity
  - Capable of hierarchic enrichment
- Physical field independent
- Applicable to 2-D, 3-D, shells, non-linear

Topics

- Laminated plate models tested:
  - Kirchhoff
  - Mindlin
  - Reddy - $C^1(\Omega)$ kinematic model
- Aspects of GFEM with arbitrary continuity
  - $C^\infty(\Omega)$ exponential functions in **convex** clouds
  - $C^k(\Omega)$ polynomial functions in **convex** clouds
  - $C^k(\Omega)$ R-functions in **non-convex** clouds
- Numerical results
  - Integrability
  - Energy convergence
  - Pointwise stresses (transverse shear)

Some key references

- Belytschko, 1994, Element Free Galerkin Method
- Duarte & Oden, 1995, 1996, hp-Cloud Method
- Babuska & Melenk, 1995, 1996, Partition of Unity Method
- Duarte, Kim & Quaresma, 2009, Arbitrarily smooth Generalized Element approximation
- Edwards, H.C., 1996, C^∞ Finite Element basis functions
- Rvachev & Sheiko, 1995, R-functions in BVP in mechanics
- Among many others

Continuity in Displacement-based FEM

Typical plate models

- Kirchhoff

$$\begin{cases} u(x,y,z) = u^o(x,y) + z\,w_{,x} \\ v(x,y,z) = v^o(x,y) + z\,w_{,y} \\ w(x,y,z) = w(x,y) \end{cases}$$

- Mindlin

$$\begin{cases} u(x,y,z) = u^o(x,y) + z\,\psi_x(x,y) \\ v(x,y,z) = v^o(x,y) + z\,\psi_y(x,y) \\ w(x,y,z) = w(x,y) \end{cases}$$

- Reddy

$$\begin{cases} u(x,y,z) = u^o + z\,\psi_x + z^3\left[\alpha\left(\psi_x + w_{,x}\right)\right] \\ v(x,y,z) = v^o + z\,\psi_y + z^3\left[\alpha\left(\psi_y + w_{,y}\right)\right] \\ w(x,y,z) = w(x,y) \end{cases}$$

GFEM – Arbitrary Continuity

C^∞

Cloud edge function

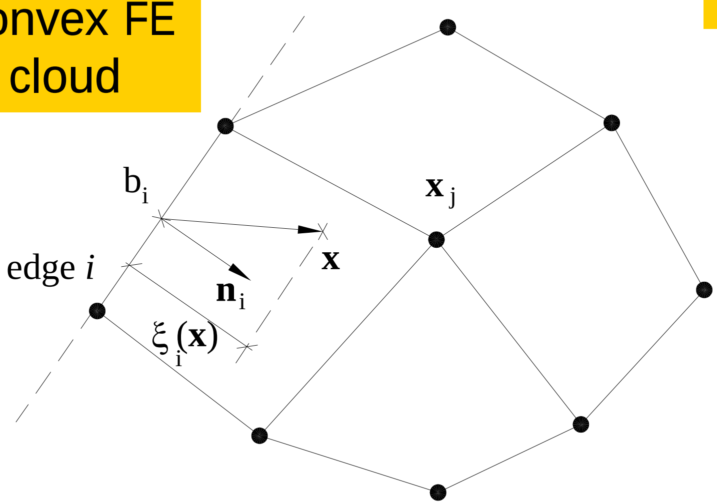
$$\varepsilon_{j,i}\left(x\right) \doteq \begin{cases} Ae^{-(\xi_i/B)^{-\gamma}}, & \xi_i(x) > 0 \\ 0, & \xi_i(x) \le 0 \end{cases}$$

$$\begin{cases} A = Exp\left[\left(\dfrac{1-2^{\gamma}}{\log_e \beta}\right)^{1/\gamma}\right] \\ B = h_{j,i}\left(\dfrac{\log_e \beta}{1-2^{\gamma}}\right)^{1/\gamma} \end{cases}$$

Weighting function

$$W_j\left(x\right) \doteq \prod_{i=1}^{M_\alpha} \varepsilon_{j,i}\left(\xi_i\right)$$

Convex FE cloud



Partition of unity

$$\phi_j\left(\boldsymbol{x}\right) = \frac{W_j\left(\boldsymbol{x}\right)}{\sum_p W_p\left(\boldsymbol{x}\right)}$$

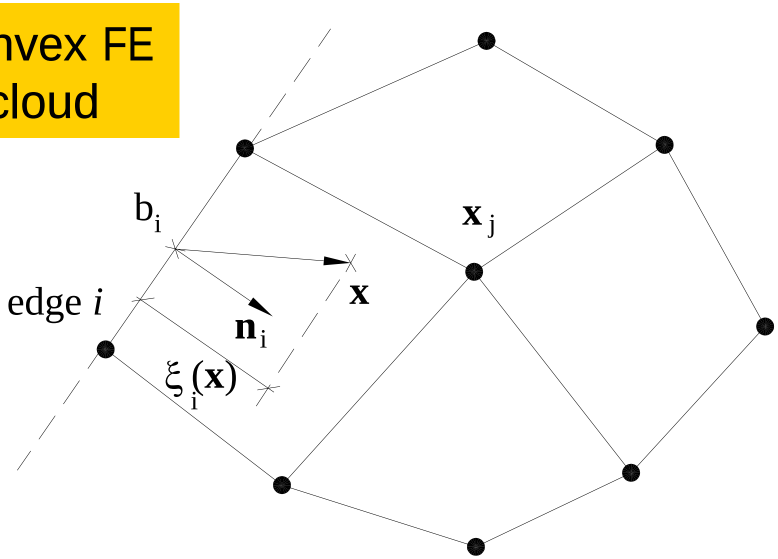
GFEM – Arbitrary Continuity

C^k

Cloud edge function - Polynomial

$$\widehat{\varepsilon}_{\alpha,j}\left[\xi_j\left(\mathbf{x}\right)\right] = \varepsilon_{\alpha,j}\left(\mathbf{x}\right) := \begin{cases} \left(\xi_j / h_j\right)^{\mathrm{P}} & if \quad 0 < \xi_j \\ 0 & \text{, otherwise} \end{cases}$$

Convex FE cloud



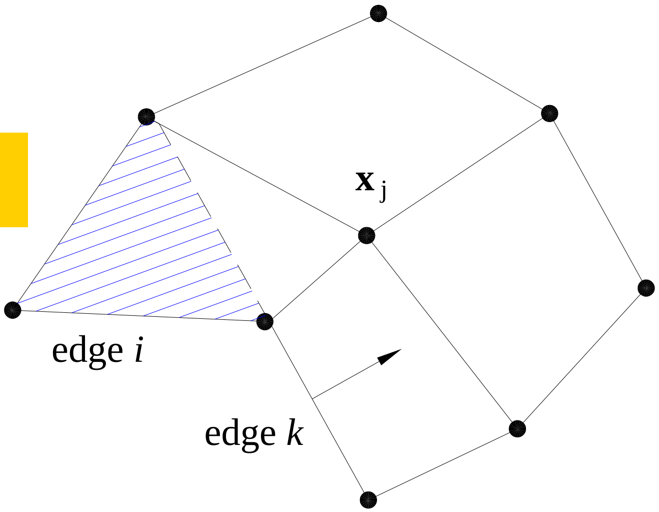
Weighting function

$$W_j\left(x\right) \doteq \prod_{i=1}^{M_\alpha} \varepsilon_{j,i}\left(\xi_i\right)$$

R (Rvachev) - Functions

C^k

Non Convex Cloud



$$\varepsilon_{j,i}(\boldsymbol{x}) \stackrel{\text{def}}{=} \begin{cases} e^{-\xi_i(\boldsymbol{x})^{-\gamma}} & , \quad \xi_j > 0 \\ 0 & , \quad \xi_j(\boldsymbol{x}) \leq 0 \end{cases} \Rightarrow \varepsilon_{j,ik}(\boldsymbol{x}) \stackrel{\text{def}}{=} \frac{\left(\varepsilon_{j,i}(\boldsymbol{x}) \vee_0^k \varepsilon_{j,k}(\boldsymbol{x})\right)}{\left(\varepsilon_{j,i}(\boldsymbol{x}_j) \vee_0^k \varepsilon_{j,k}(\boldsymbol{x}_j)\right)}$$

Boolean R-Functions

C^k

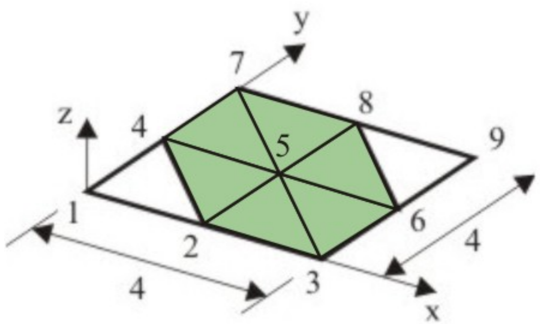
$$(x \vee_0^k y) \stackrel{\text{def}}{=} \left(x + y + \sqrt{x^2 + y^2}\right)\left(x^2 + y^2\right)^{k/2}$$

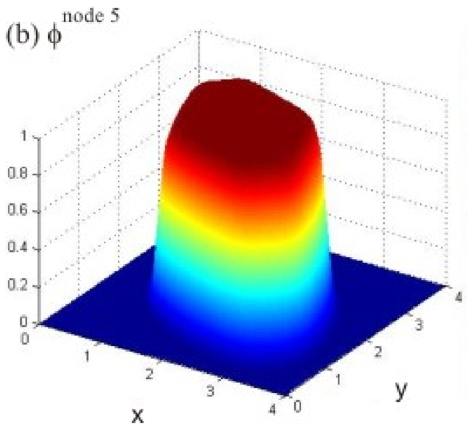
R-Function

Analytic except at: $(x = 0 \,, y = 0)$ – k times differentiable

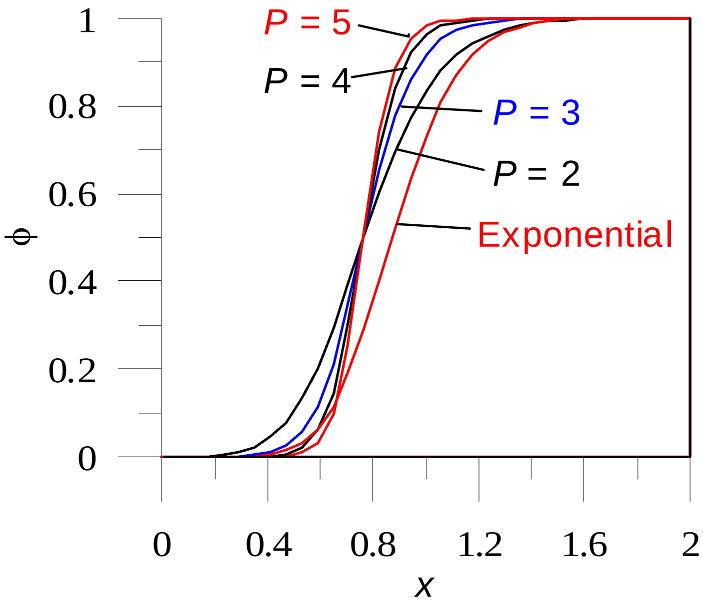
$$(x \vee_0^k y) \begin{cases} = 0 \Leftrightarrow x = 0 \text{ e } y = 0 \\ > 0 \;\forall\; x > 0 \text{ ou } y > 0 \end{cases}$$

Convex cloud

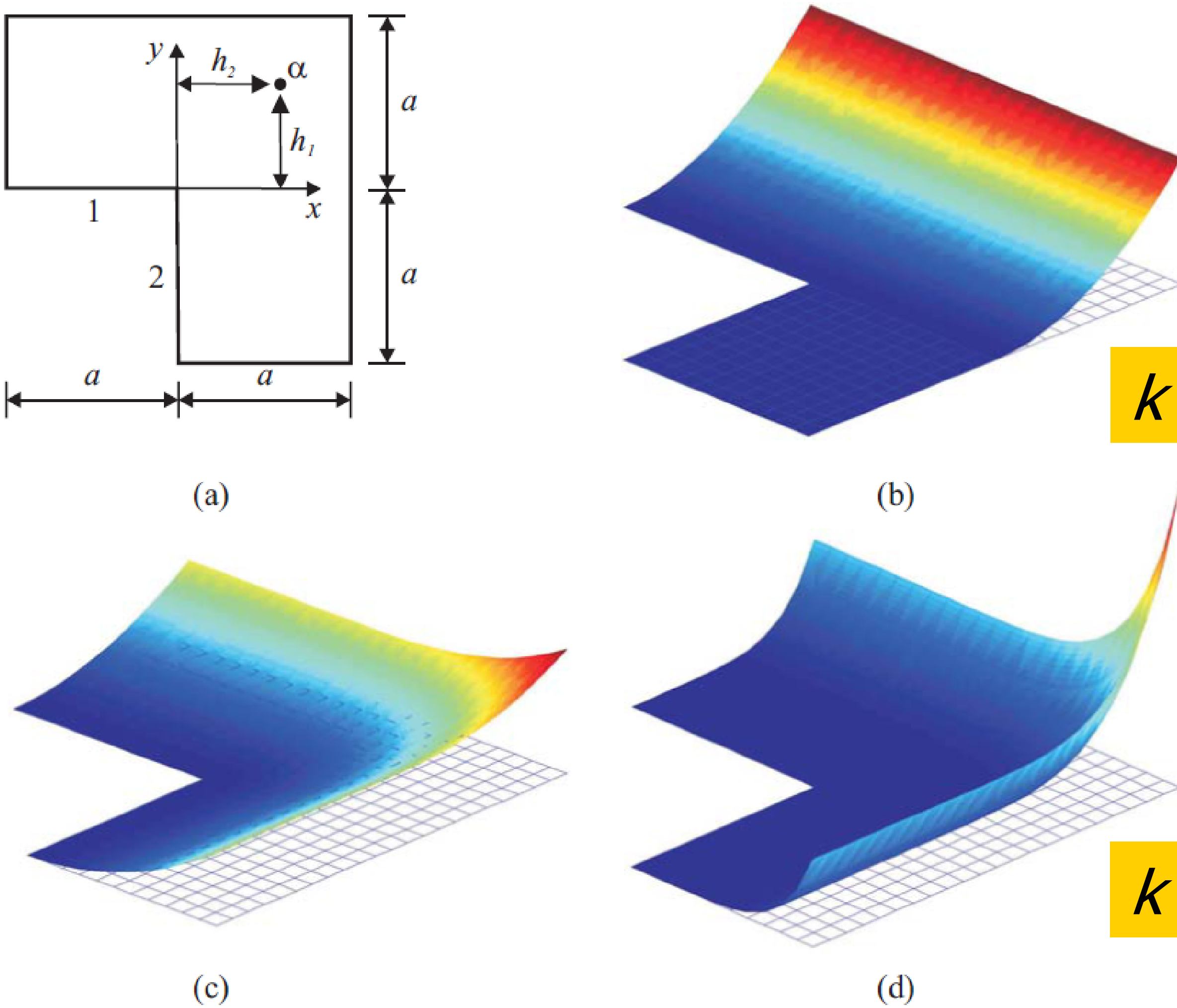




Partition of unity along y = 2



Non-convex cloud – Polynomial edge function Edges 1 and 2

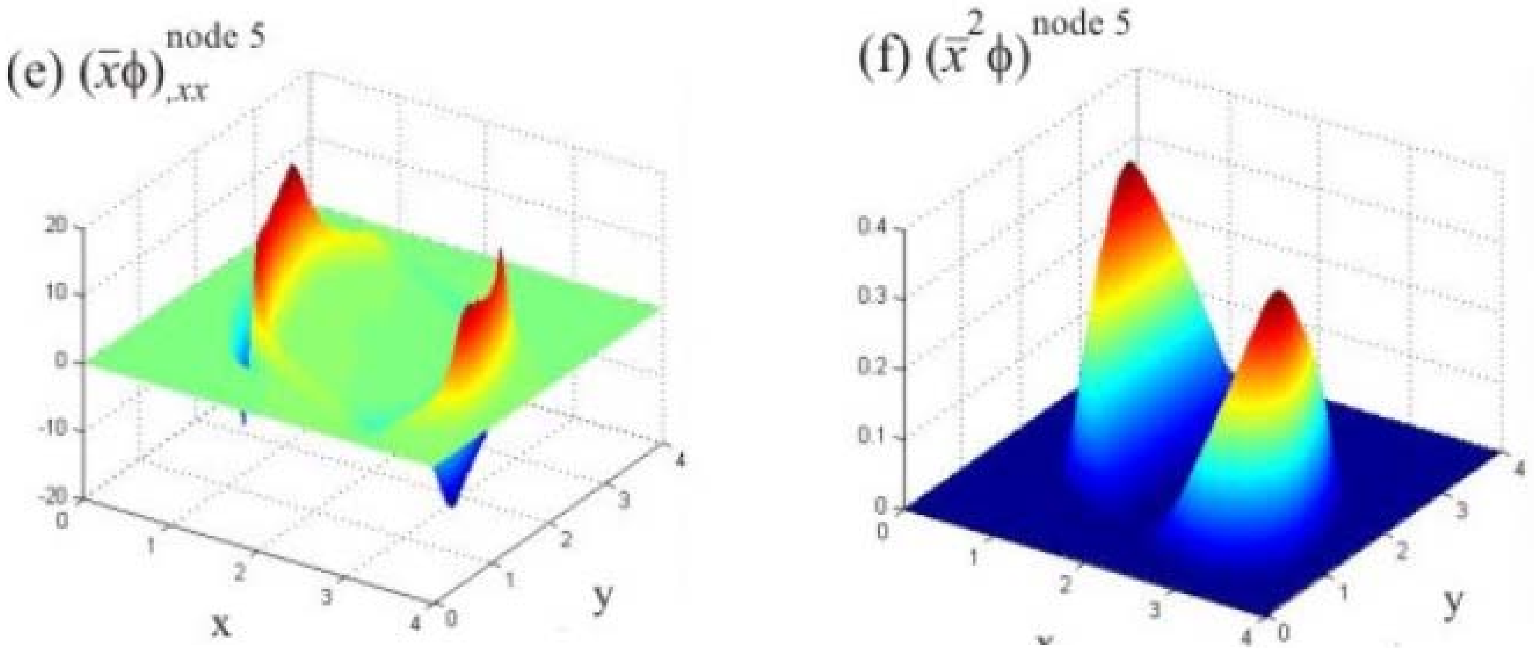


$k = 0$

$k = 0$

$k = 2$

Enriched functions



– linear basis: $\{1, (x - x_\alpha), (y - y_\alpha)\}$ or

– quadratic basis: $\left\{1, (x - x_\alpha), (y - y_\alpha), (x - x_\alpha)^2 \right.$
$\left. (x - x_\alpha)(y - y_\alpha)(y - y_\alpha)^2 \right\}.$

Laminated plate functionals

- Kirchhoff
- Mindlin
- Reddy

$$G(\mathbf{d},\delta\mathbf{d}) = \int_\Omega \begin{Bmatrix} \delta\boldsymbol{\varepsilon}^o \\ \delta\boldsymbol{\kappa} \\ \delta\boldsymbol{\kappa}_3 \end{Bmatrix}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{L} \\ \mathbf{B} & \mathbf{D} & \mathbf{F} \\ \mathbf{L} & \mathbf{F} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^o \\ \boldsymbol{\kappa} \\ \boldsymbol{\kappa}_3 \end{Bmatrix} d\Omega$$

$$+ \int_\Omega \begin{Bmatrix} \delta\gamma_c \\ \delta\boldsymbol{\kappa}_c \end{Bmatrix}^T \begin{bmatrix} \mathbf{A_C} & \mathbf{D_C} \\ \mathbf{D_C} & \mathbf{F_C} \end{bmatrix} \begin{Bmatrix} \gamma_c \\ \boldsymbol{\kappa}_c \end{Bmatrix} d\Omega$$

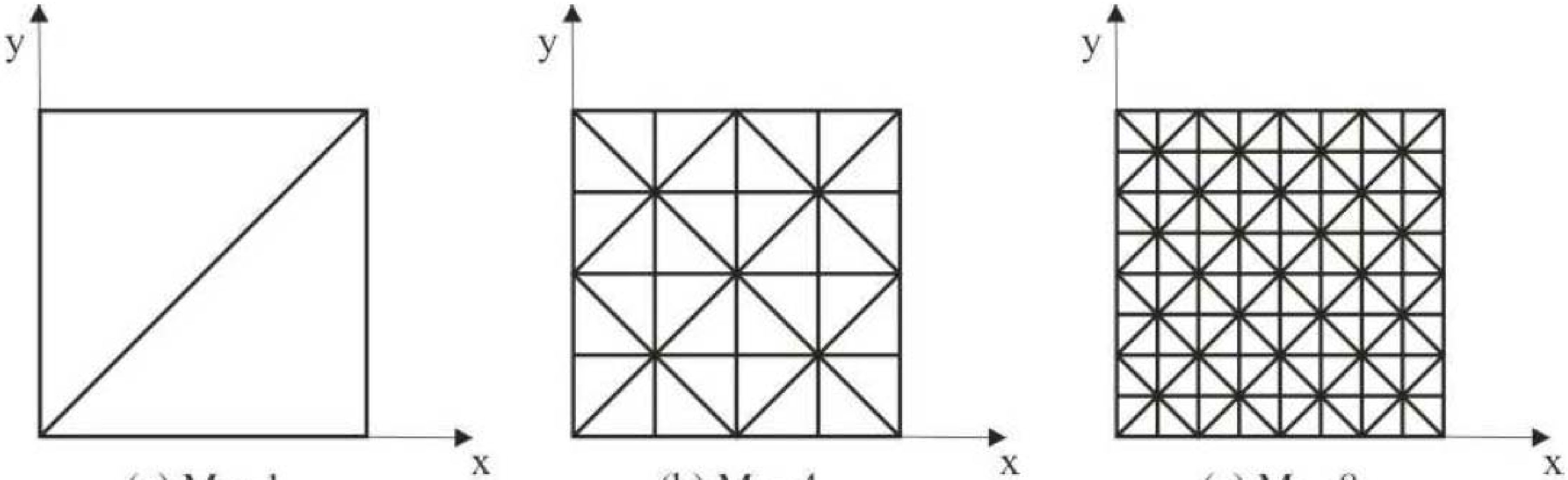
$$l(\delta w) = \int_\Omega \delta w \; q \; d\Omega$$

Base problem

- 3 orthotropic layers
- [0/90/0]
- Sinusoidal transverse load

$E_1 = 175$ GPa $\quad G_{12} = 3.5$ GPa

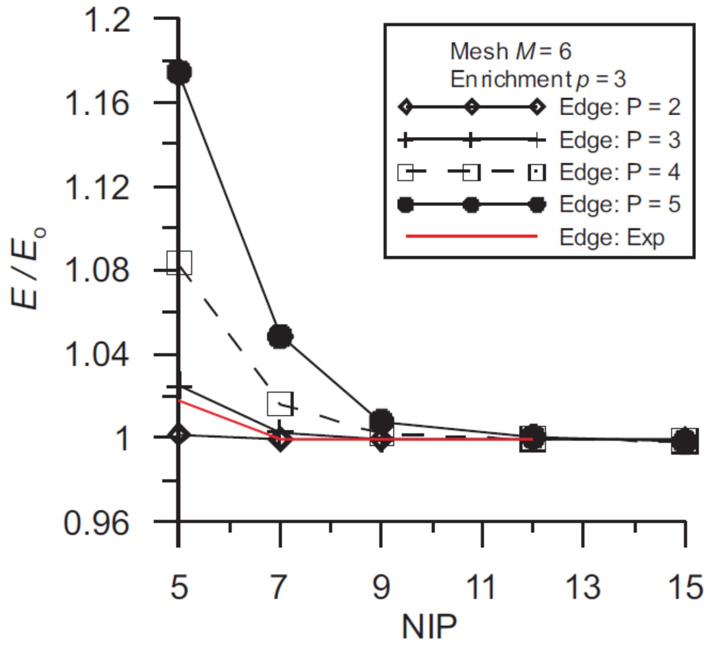
$E_2 = 7$ GPa $\quad \nu_{12} = 0.25$

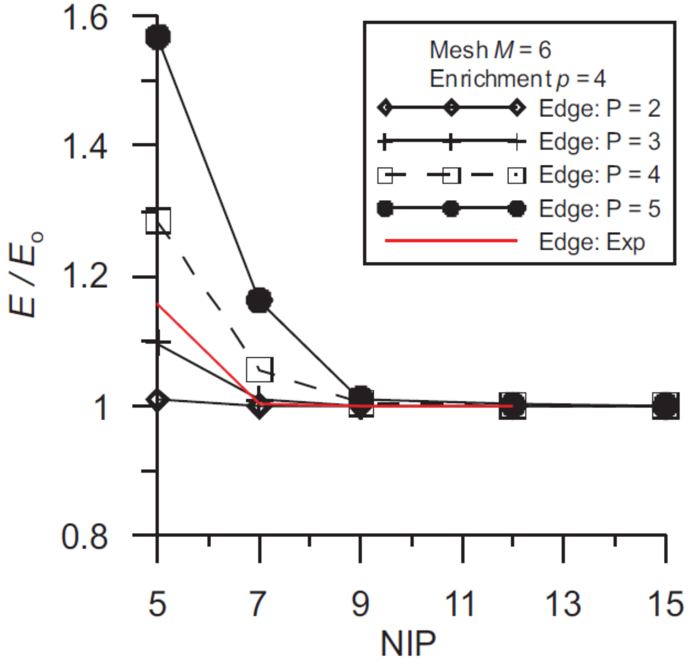


(a) M = 1 (b) M = 4 (c) M = 8

Kirchhoff model

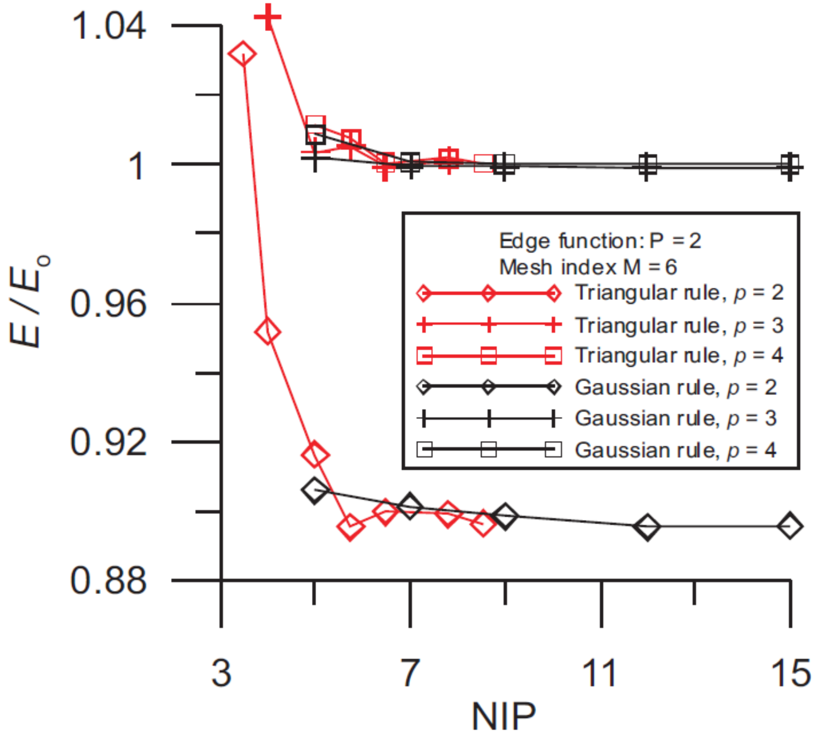
- Integrability – Gaussian integration

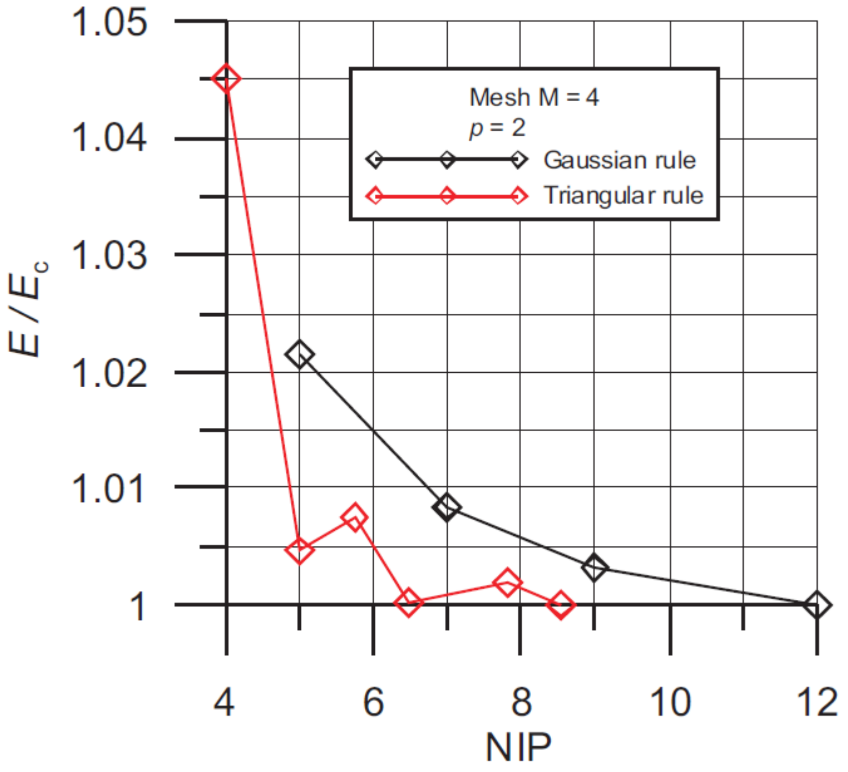




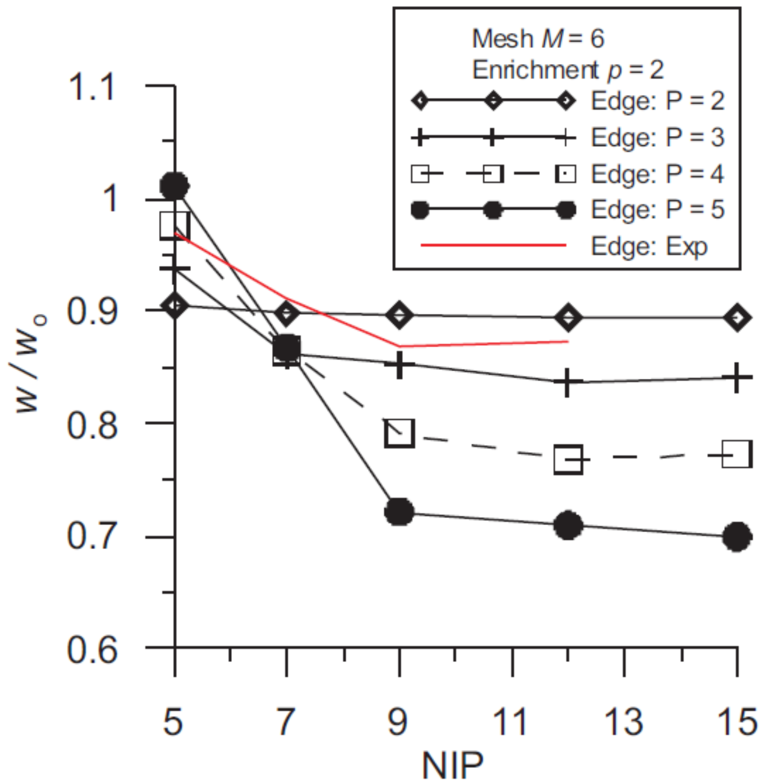
Kirchhoff model

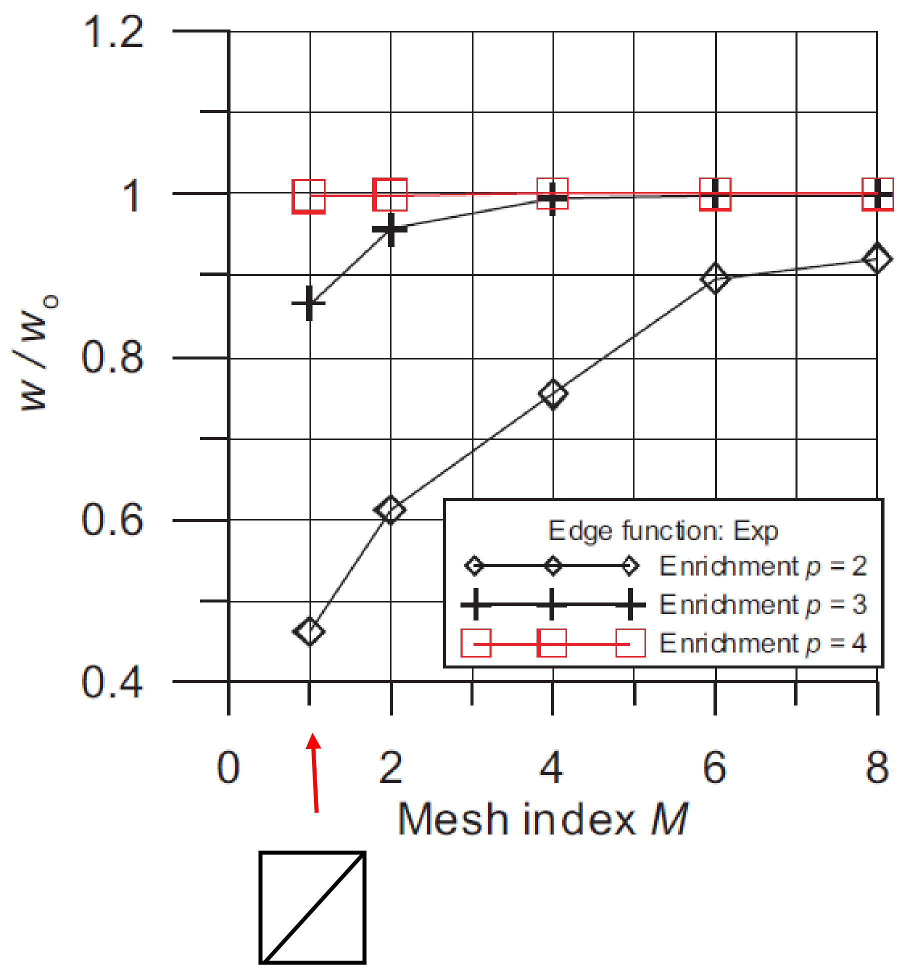
Gaussian and triangular integration rules





Kirchhoff model - Displacement



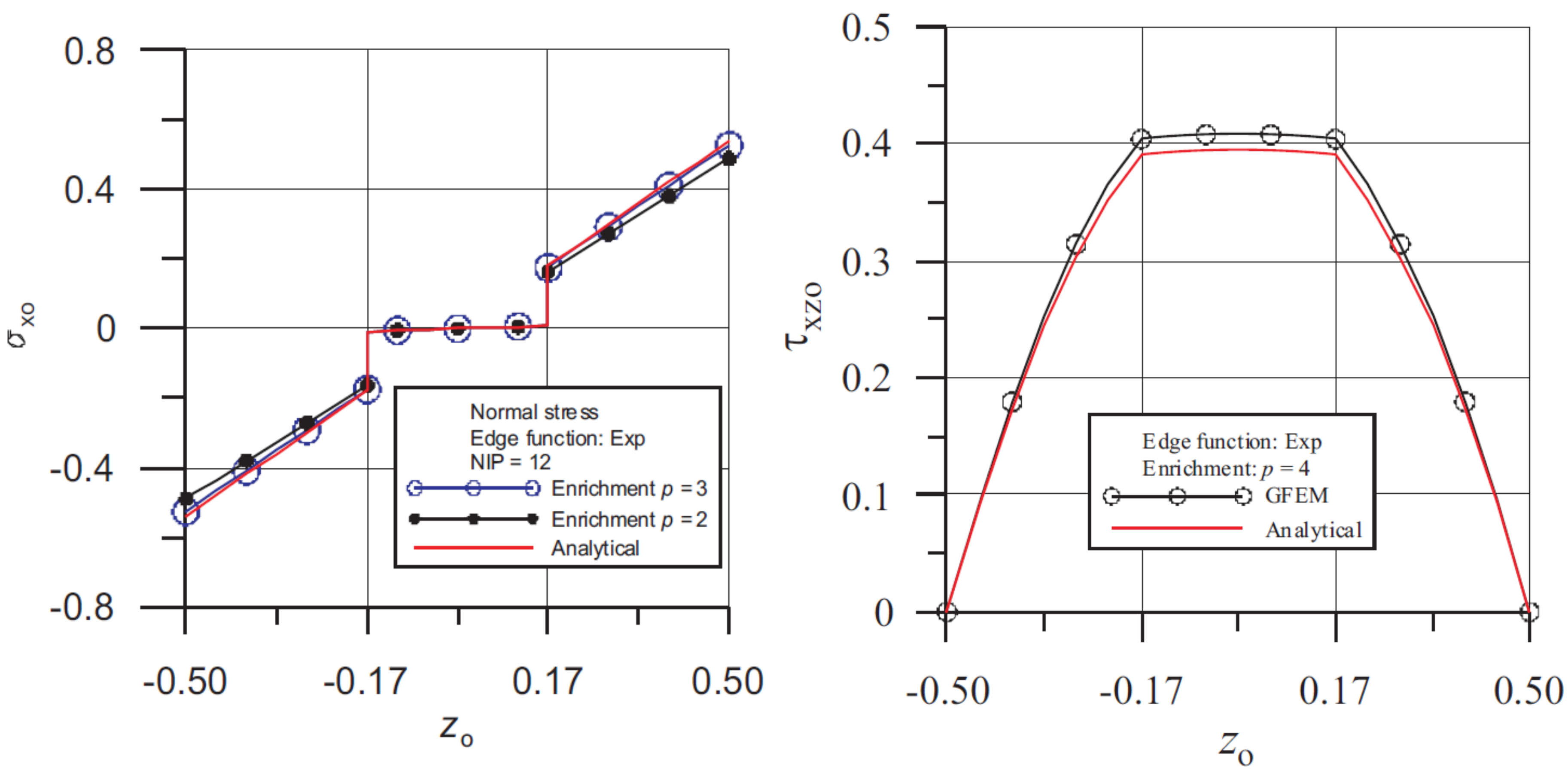


Transverse shear stresses – by integration

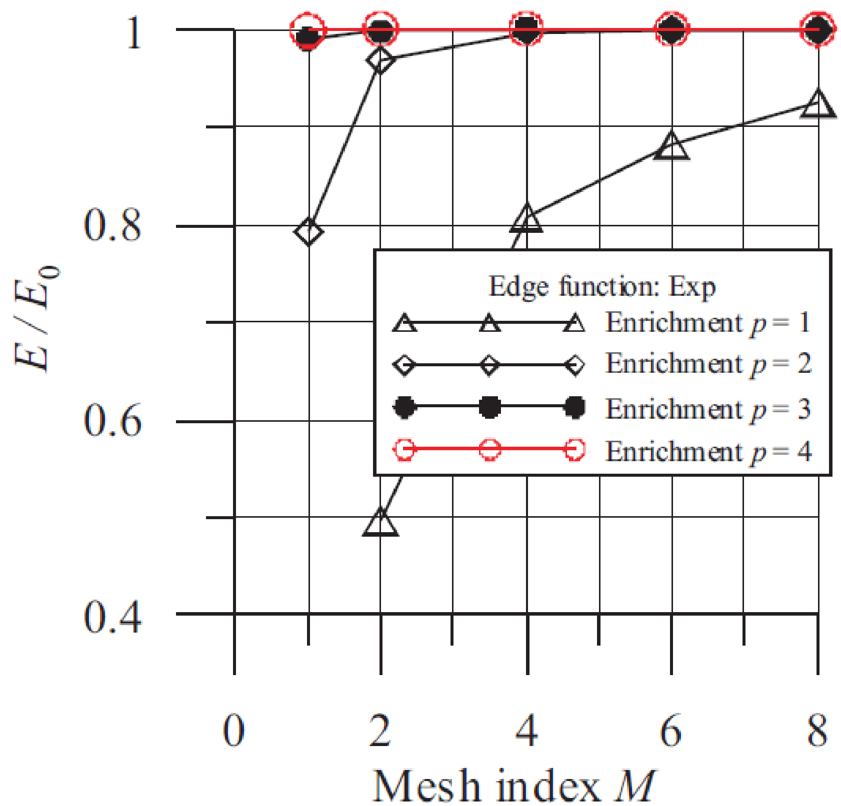
$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} = 0 \end{cases} \longrightarrow \begin{cases} \dfrac{\partial \boldsymbol{\sigma}}{\partial x} = \overline{\boldsymbol{Q}} \left[\dfrac{\partial \boldsymbol{\varepsilon}^o}{\partial x} + z \dfrac{\partial \boldsymbol{\kappa}}{\partial x} + z^3 \dfrac{\partial \boldsymbol{\kappa}_3}{\partial x} \right] \\ \dfrac{\partial \boldsymbol{\sigma}}{\partial y} = \overline{\boldsymbol{Q}} \left[\dfrac{\partial \boldsymbol{\varepsilon}^o}{\partial y} + z \dfrac{\partial \boldsymbol{\kappa}}{\partial y} + z^3 \dfrac{\partial \boldsymbol{\kappa}_3}{\partial y} \right] \end{cases}$$

$$\tau_{xz}^k(\boldsymbol{x}, z) = \tau_{xz}^k(\boldsymbol{x}, z_{k-1}) - \int_{z_{k-1}}^{z} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) dz$$

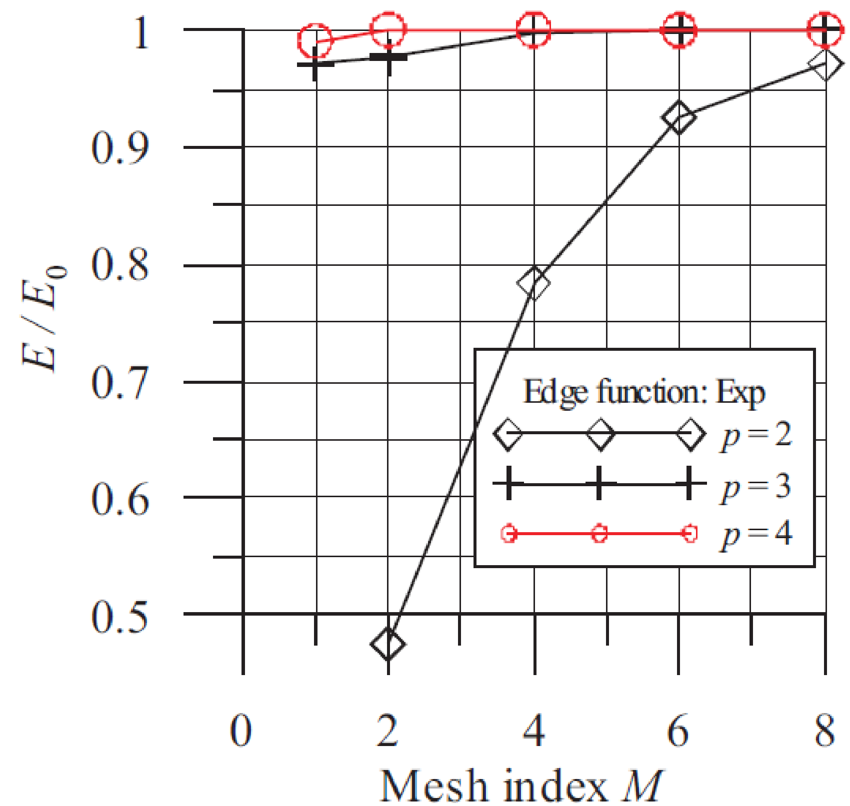
Kirchhoff model - Stresses



Mindlin Model – *h* -convergence

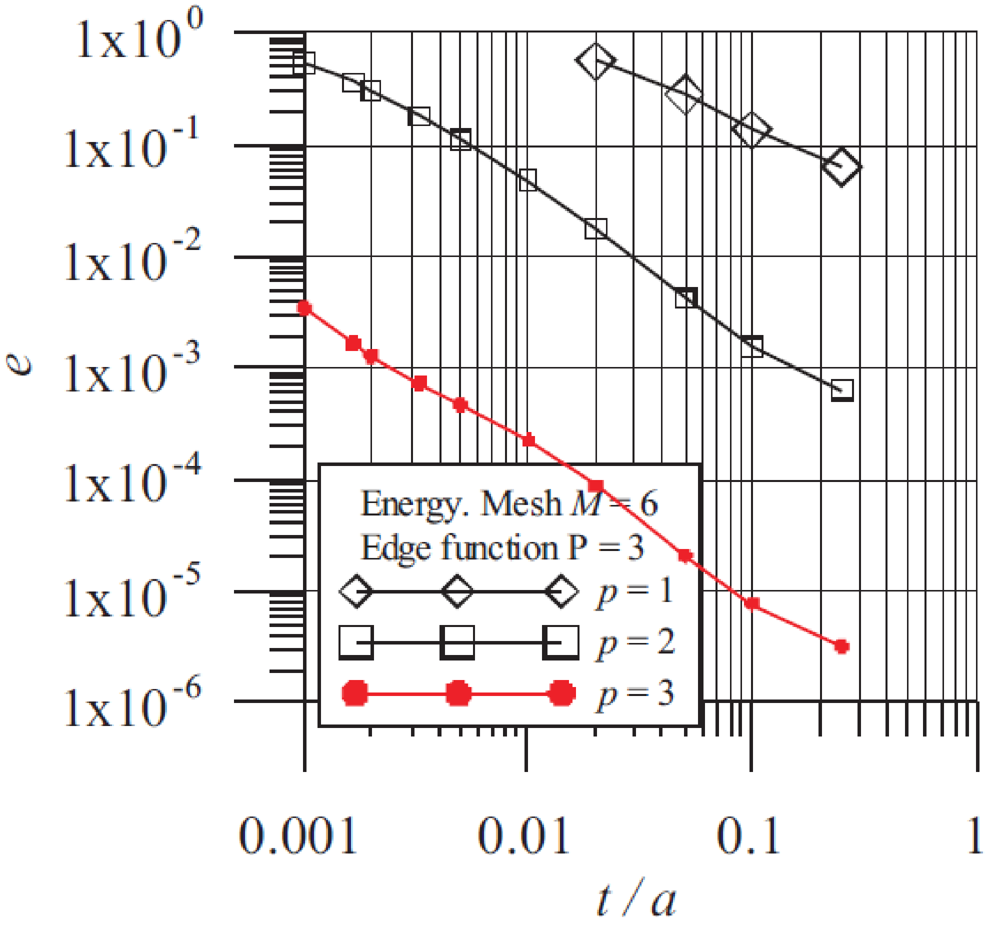


a/ *t* = 4

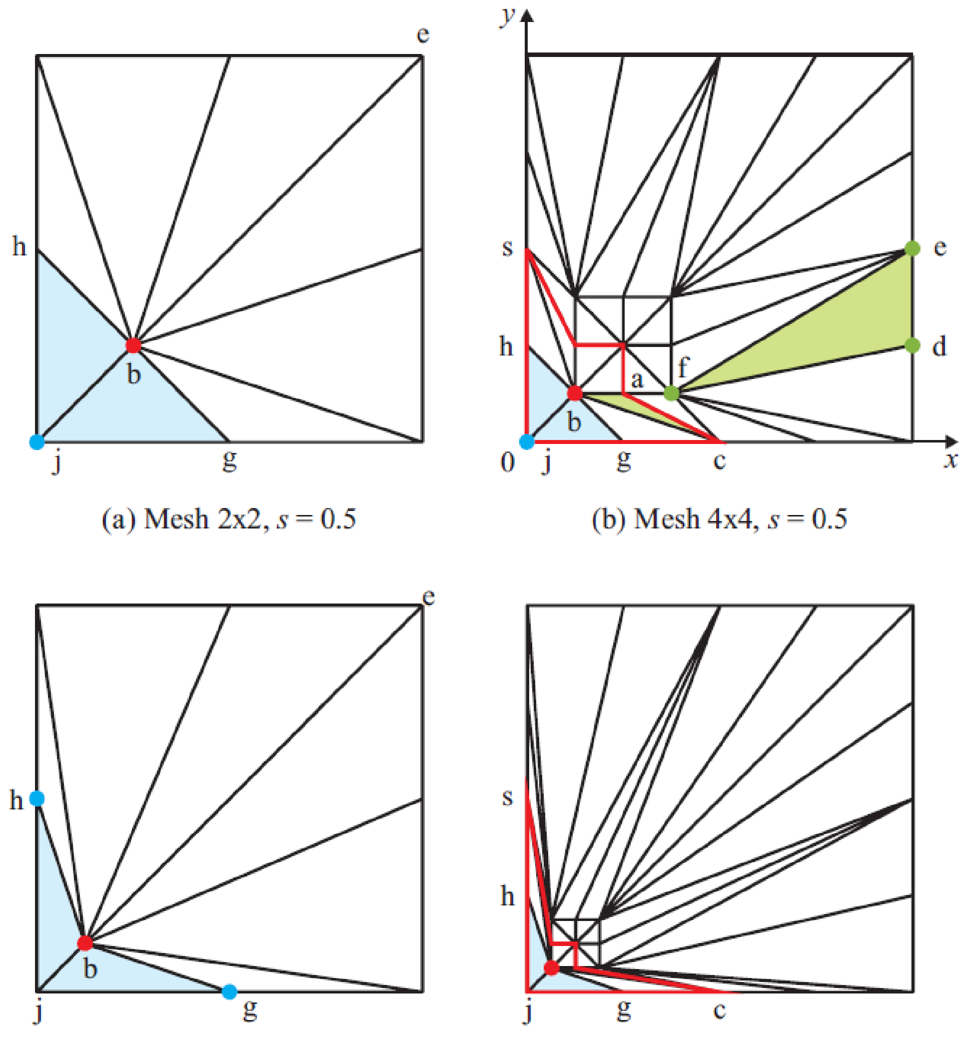


a/ *t* = 100

Mindlin Model – Thickness effect



Mesh distortion



$$r_m\big|_{abc} = \frac{base}{heigth}$$

$$r_m = 9802$$

$$m_m = \frac{A_{def}}{A_{abc}}$$

$$m_m = 4950$$

Mesh distortion

$$r_m\big|_{abc} = \frac{base}{height}$$

$$r_m = 9802$$

$$m_m = \frac{A_{def}}{A_{abc}}$$

$$m_m = 4950$$

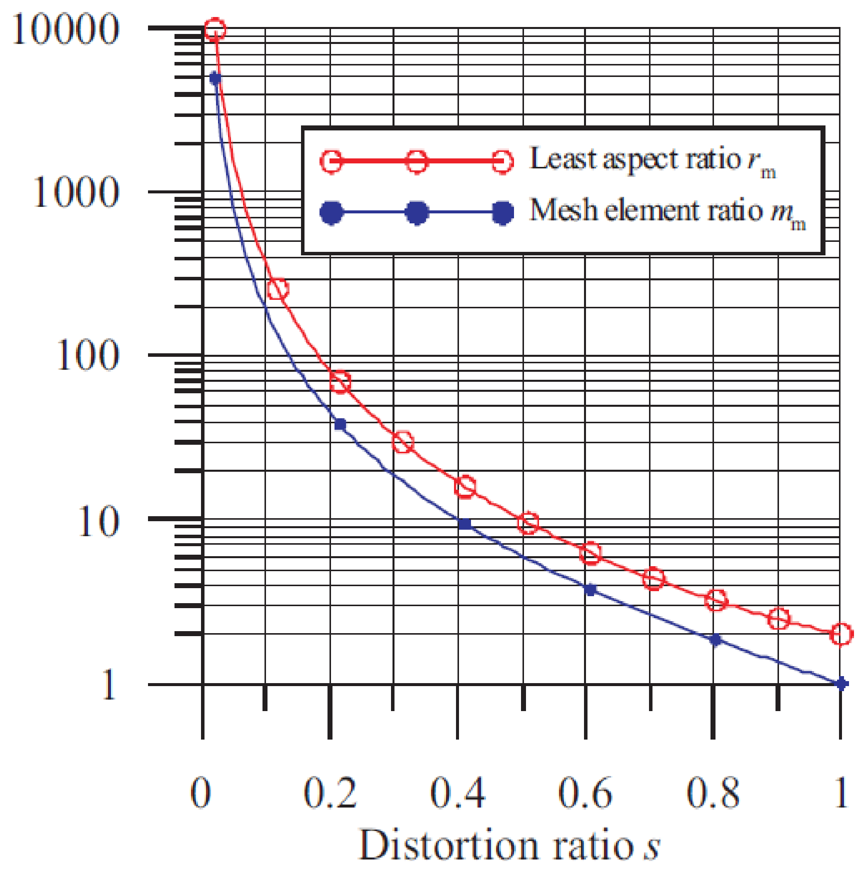
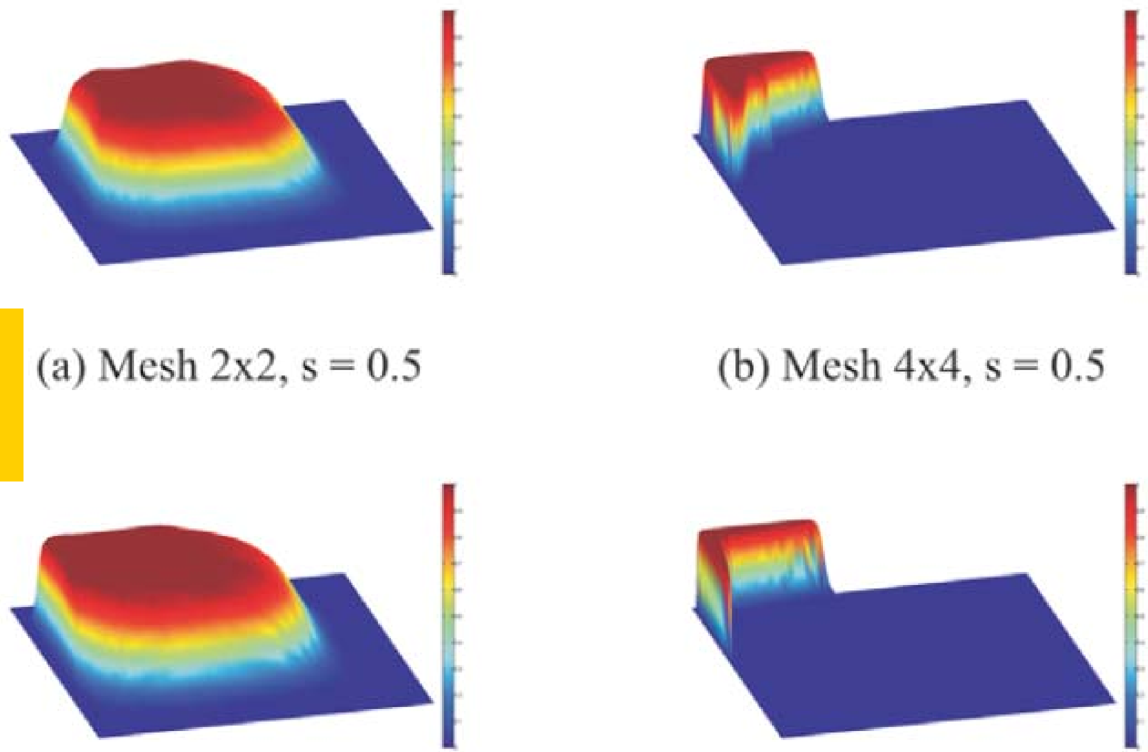


Figure 19: Variation of element aspect raio and element size ratio with distortion ratio for distorted mesh $M = 2$.

Mesh distortion

Convex cloud

Non-convex cloud



(a) Mesh 2x2, s = 0.5

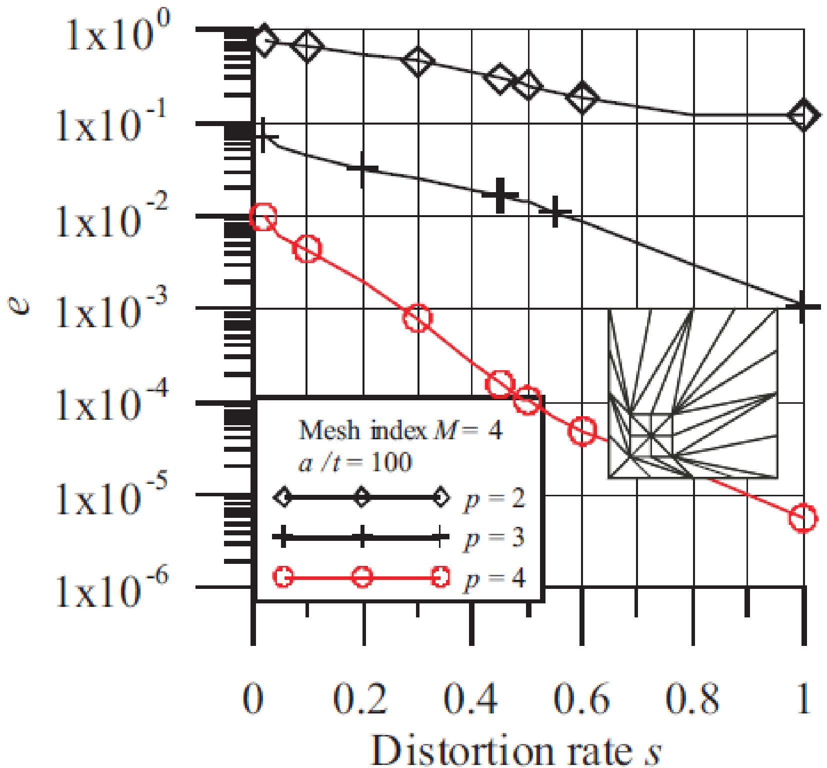
(b) Mesh 4x4, s = 0.5

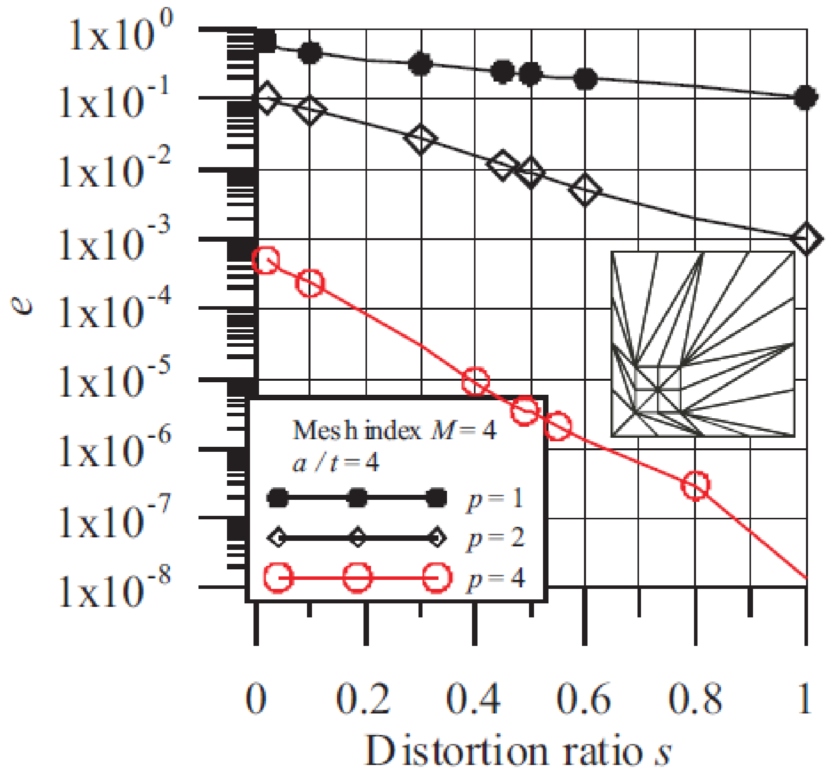
(c) Mesh 2x2, s = 0.25

(d) Mesh 4x4, s = 0.25

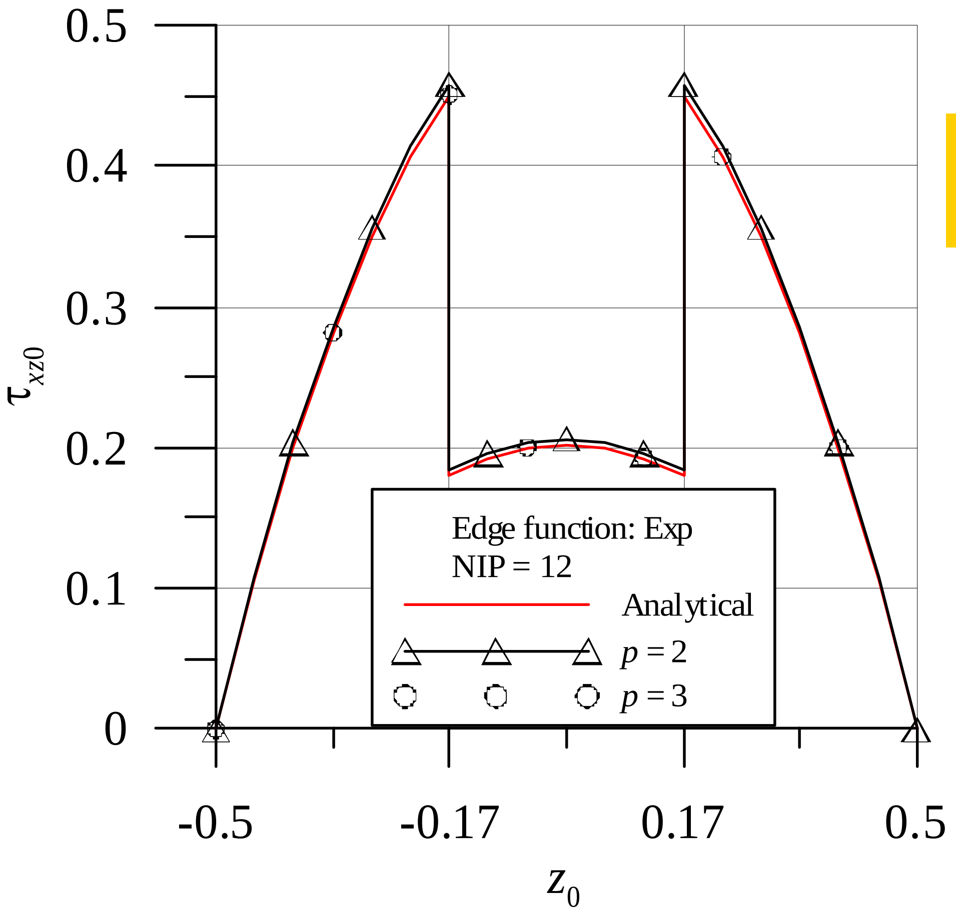
Figure 14: Views of non-enriched basis functions defined in convex and non-convex edges.

Mindlin Model – Mesh distortion



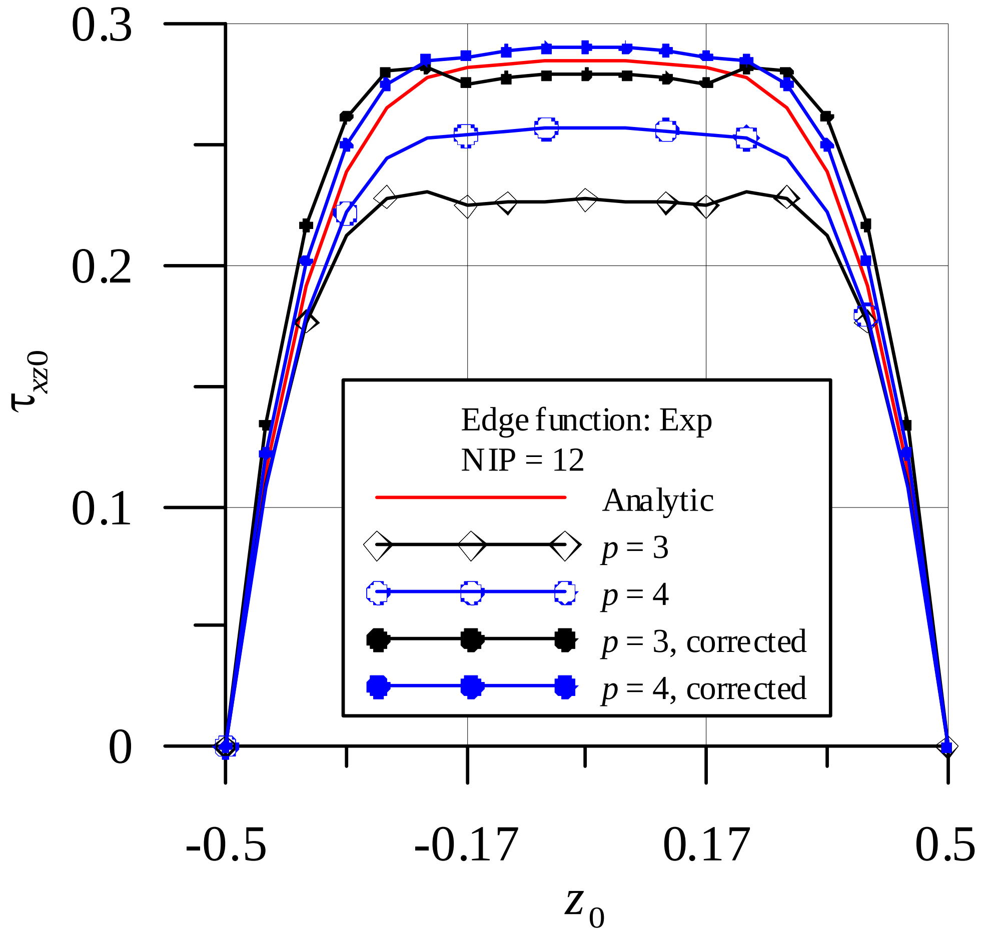


Reddy Model – Transverse shear stresses



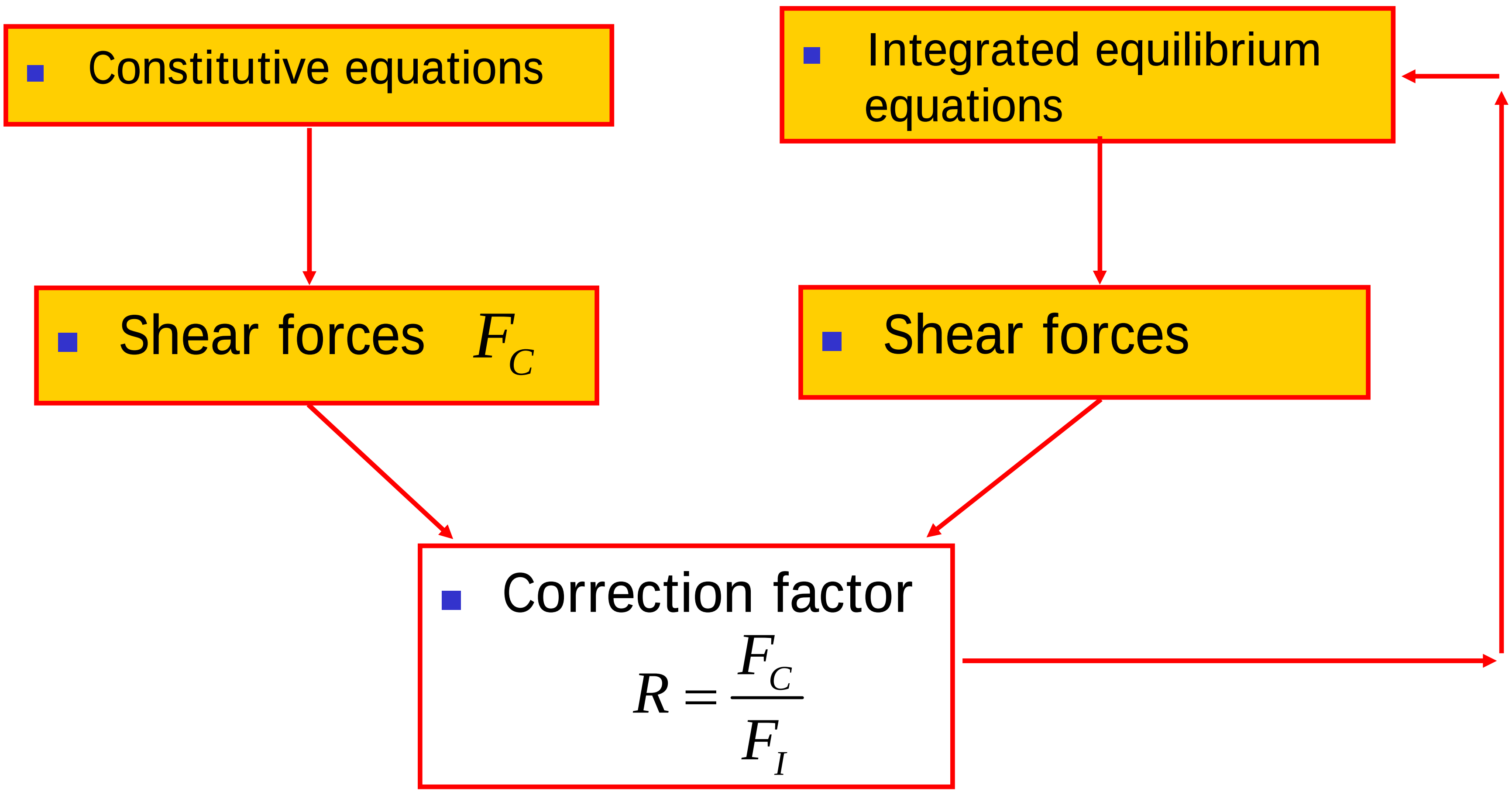
- From constitutive equations

Reddy Model – Transverse shear stresses

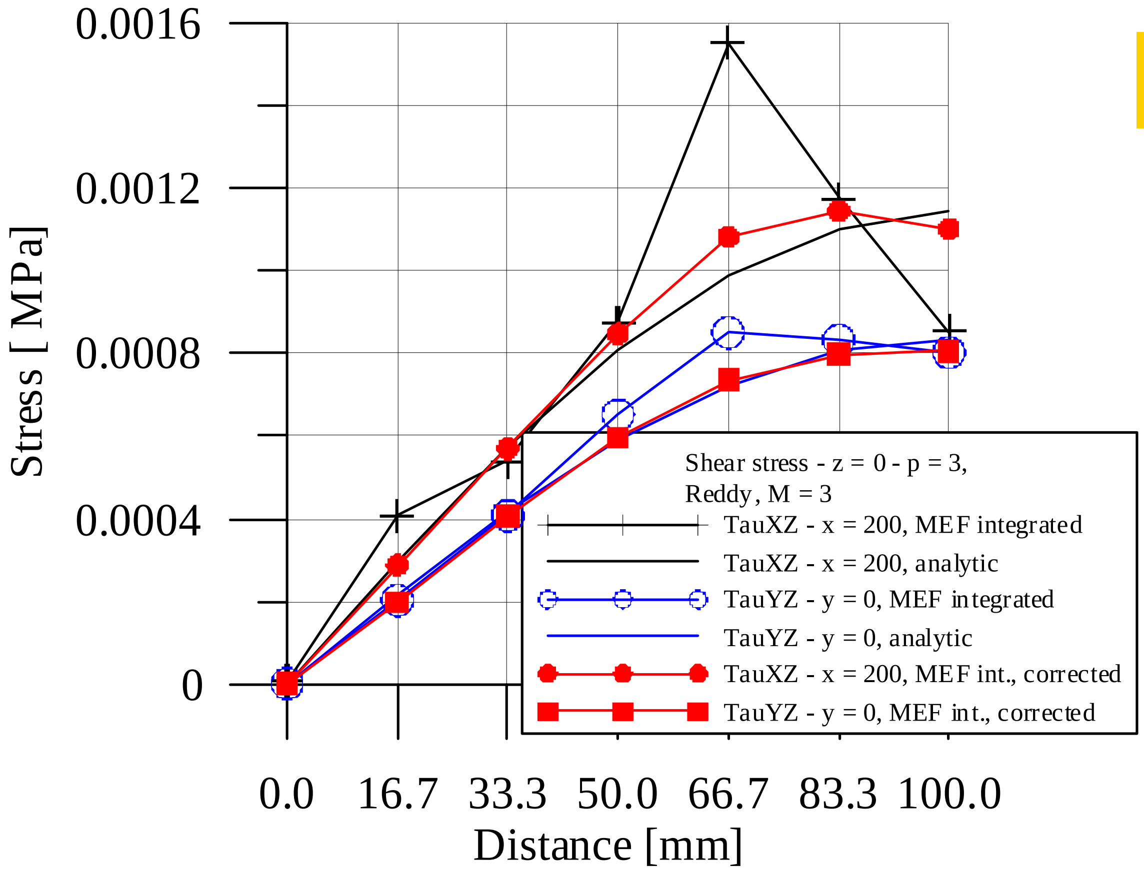


- Integrated equilibrium equations
- Corrected stresses

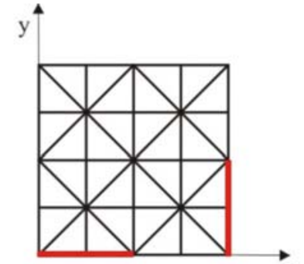
Reddy and Mindlin Models - Correction of transverse shear stresses



Reddy Model – Transverse shear stresses



Enrichment *p* = 3



(b) M = 4

GFEM – with arbitrary Continuity

- Good aspects observed
  - Ease and precise stress determination
    - including transverse inter-layer stresses
  - Excelent behavior in high mesh distortion
    - The approximation functions defined in global coordinates
  - Polynomial edge functions of low degrees are efficient compared to the exponential ones

- Requires attention
  - Integration effort