Thematic Conference

XFEM, GFEM and fictitious domain methods: recent developments and applications.

XFEM 2013

An IACM special interest conference

11-13 September 2013, Lyon, France

Benefits provided by partitions of unity with high regularity in crack modeling through enrichment procedures

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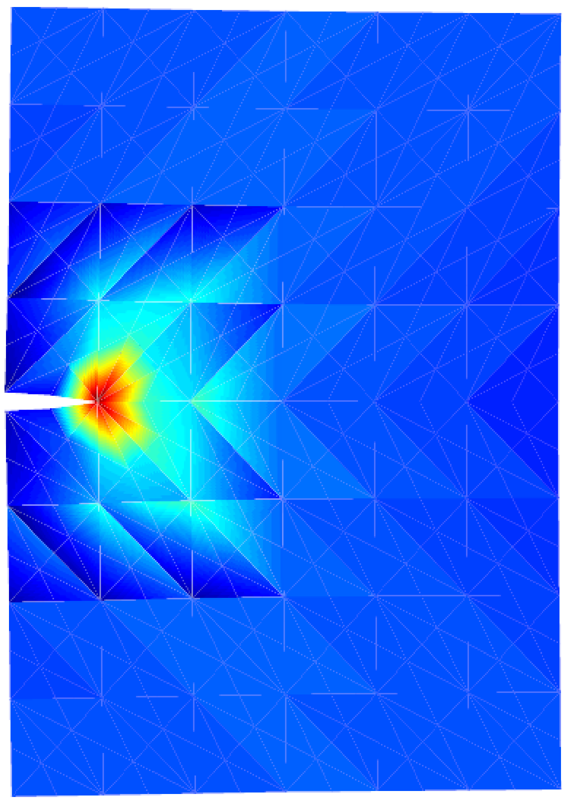
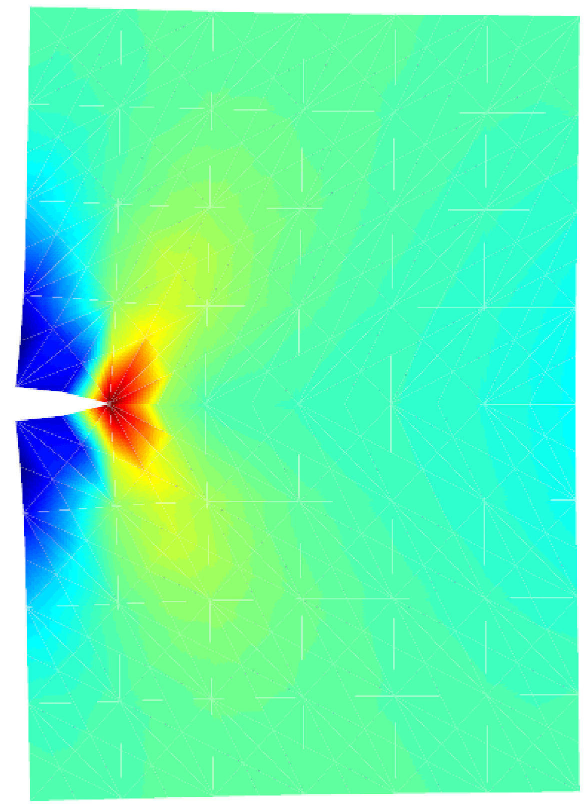
Motivation



GFEM-C^0

?



GFEM-C^k

Presentation topics

- Continuous partition of unity with C^k-GFEM
- Defining an approximation subspace
- Enrichment patterns and convergence rates
- Quality assessment through global measures
- Configurational forces method
- Quality assessment through local measures
- Smoothness, enrichments and conditioning
- Some improvements beyond...

C^∞ partition of unity – convex clouds

- No shape restrictions
- no coordinate mapping
- flat-top property
- simple numerical integration
- blending
-
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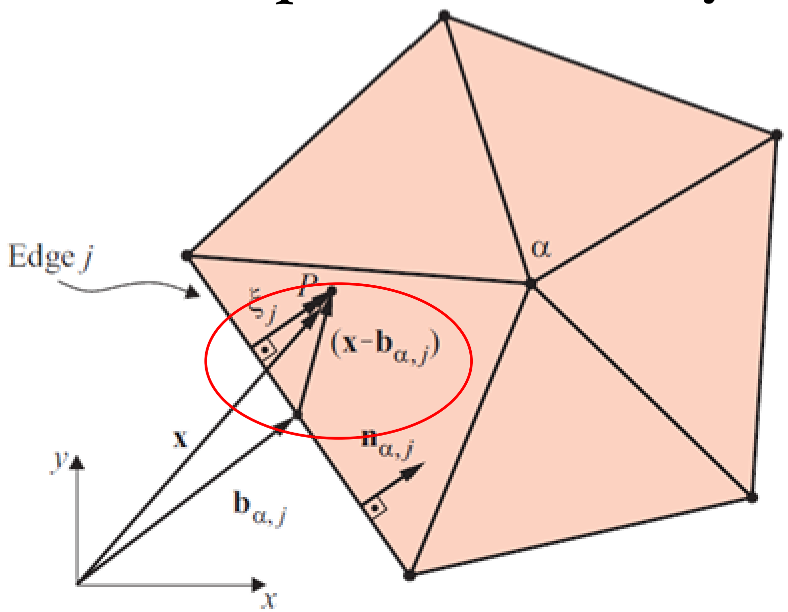
Edwards, **C^∞ finite element basis functions**, Report 45,
Institute for Computational Engineering and Sciences – The University of Texas at Austin, 2006

C^k partition of unity – convex clouds



No restriction of patch shape
Free of coordinate mapping!

$$\xi_j(\boldsymbol{x}) = \boldsymbol{n}_{\alpha,j} \cdot (\boldsymbol{x} - \boldsymbol{b}_{\alpha,j})$$

$$\varepsilon_{\alpha,j}[\xi_j(\mathbf{x})] = \widehat{\varepsilon}_{\alpha,j}(\mathbf{x}) := \begin{cases} e^{-\xi_j^{-\gamma}} & \text{if } 0 < \xi_j \\ 0, & \text{otherwise} \end{cases}$$ k = ∞

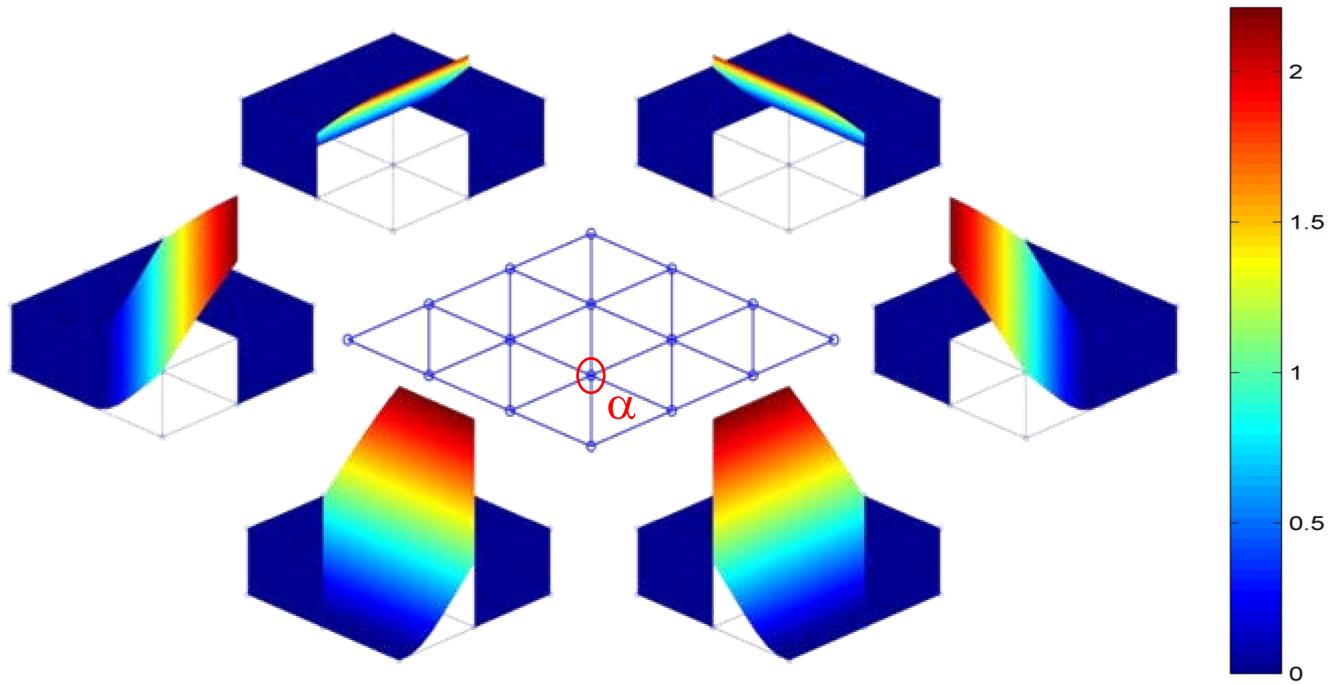
$$\varepsilon_{\alpha,j}[\xi_j(\mathbf{x})] = \widehat{\varepsilon}_{\alpha,j}(\mathbf{x}) := \begin{cases} (\xi_j / h_j)^{\mathrm{P}} & \text{if } 0 < \xi_j \\ 0, & \text{otherwise} \end{cases}$$ k = p -1

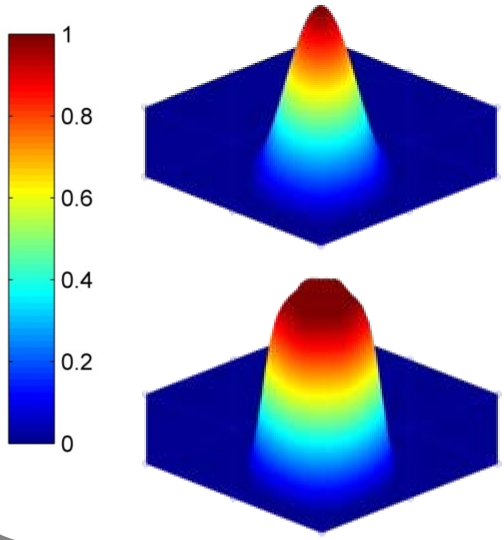
Edwards, ***C^∞ finite element basis functions***, Report 45,
Institute for Computational Engineering and Sciences – The University of Texas at Austin, 1996

Duarte, Kim and Quaresma, ***Arbitrarily smooth generalized finite element approximations***. Computer Methods in Applied Mechanics and Engineering, 196 (2006)

Barcellos, Mendonça and Duarte, ***A Ck continuous generalized finite element formulation applied to laminated Kirchhoff plate model***. Computational Mechanics, 44 (2009)

C^∞ partition of unity – convex clouds





$$\mathcal{W}_\alpha(\boldsymbol{x}) := \prod_{j=1}^{M_\alpha} \varepsilon_{\alpha,j}(\xi_j) \qquad M_\alpha = 6$$

$$\varphi_\alpha(\boldsymbol{x}) = \frac{\mathcal{W}_\alpha(\boldsymbol{x})}{\sum_{\beta(\boldsymbol{x})} \mathcal{W}_\beta(\boldsymbol{x})},$$

$$\beta(\boldsymbol{x}) \in \{\gamma \mid \mathcal{W}_\gamma(\boldsymbol{x}) \neq 0\}$$

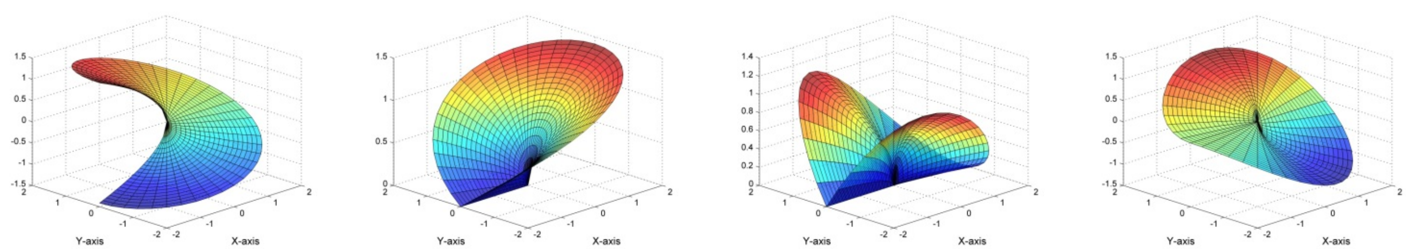
Galerkin aproximation

$$\boldsymbol{u}_p(\boldsymbol{x}) = \sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x}) \left\{ u_\alpha + \sum_{i=1}^{q_\alpha} \mathcal{L}_{\alpha i}(\boldsymbol{x}) b_{\alpha i} + \sum_{j=1}^{q_\alpha^s} \mathcal{L}_{\alpha j}^s b_{\alpha j}^s \right\}$$

if p=3 $\quad \mathcal{L}_{\alpha 9}(x,y) = \left\{ \overline{x}, \overline{y}, \overline{x}^2, \overline{x}\,\overline{y}, \overline{y}^2, \overline{x}^3, \overline{x}^2\,\overline{y}, \overline{x}\,\overline{y}^2, \overline{y}^3 \right\}$

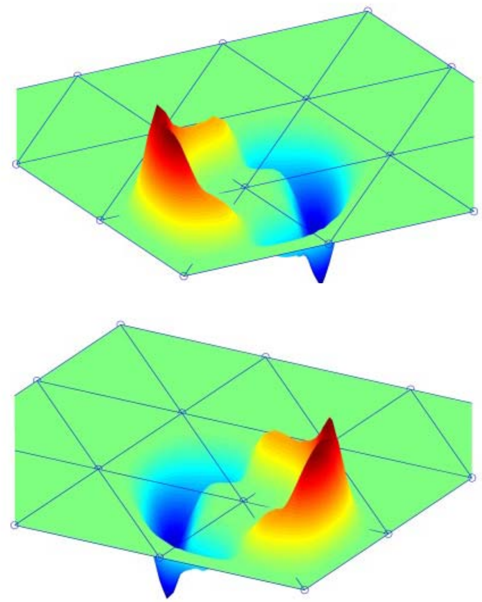
e.g. $\quad \overline{x} := \dfrac{(x - x_\alpha)}{h_\alpha}$ $\quad$ for reducing mesh dependences

$q_\alpha^s \;=\; 4 \text{ or } 0$



$$\mathcal{L}_{\alpha 4}^s(r,\theta) = \left\{ \sqrt{r}\sin\left(\frac{\theta}{2}\right), \sqrt{r}\cos\left(\frac{\theta}{2}\right), \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right\}$$

Defining the degree of an approximation



$b = p+1$ for C^0 PoU (conventional tent FEM shape function)

$b = p$ for C^k PoU

b = degree of reproducible polynomial
p = degree of polynomial enrichment

Mendonça, Barcellos and Torres, ***Robust Ck/C0 generalized FEM approximations for higher-order conformity requirements: application to Reddy's HSDT model for anisotropic laminated plates***. Composite Structures, 96 (2013)

Mendonça, Barcellos and Torres, ***Analysis of anisotropic Mindlin plate model by continuous and non-continuous GFEM***. Finite Element in Analysis and Design, 47 (2011)

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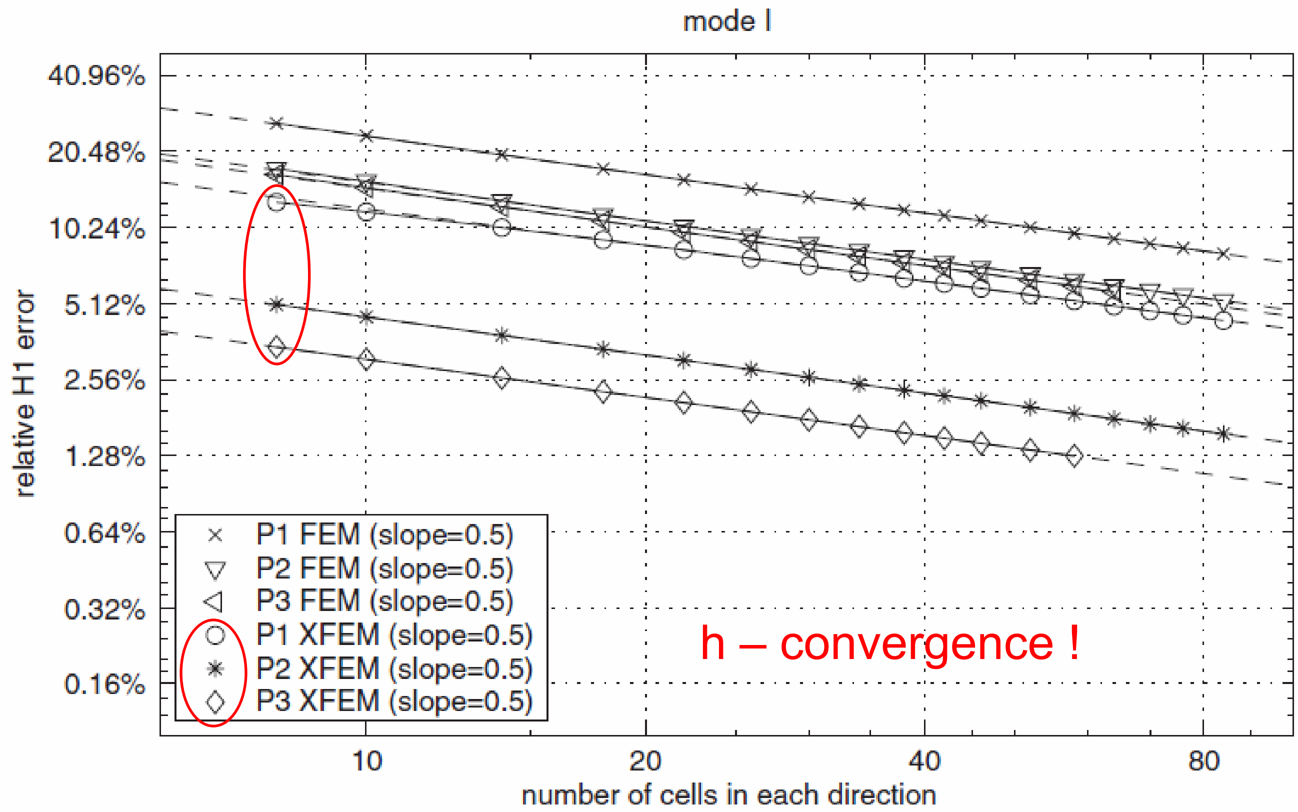
Enrichment pattern X convergence rates

Topologic enrichement



Béchet, Minnebo, Moës and Burgardt, ***Improved implementation and robustness study of the XFEM for stress analysis around cracks.*** International Journal for numerical method in engineering, 64 (2005)

Laborde, Pommier, Renard and Salaun, ***High-order extended finite element method for cracked domains.*** International Journal for numerical method in engineering, 64 (2005)

Continuous partition of unity with C^k-GFEM

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Quality assessment through global measures

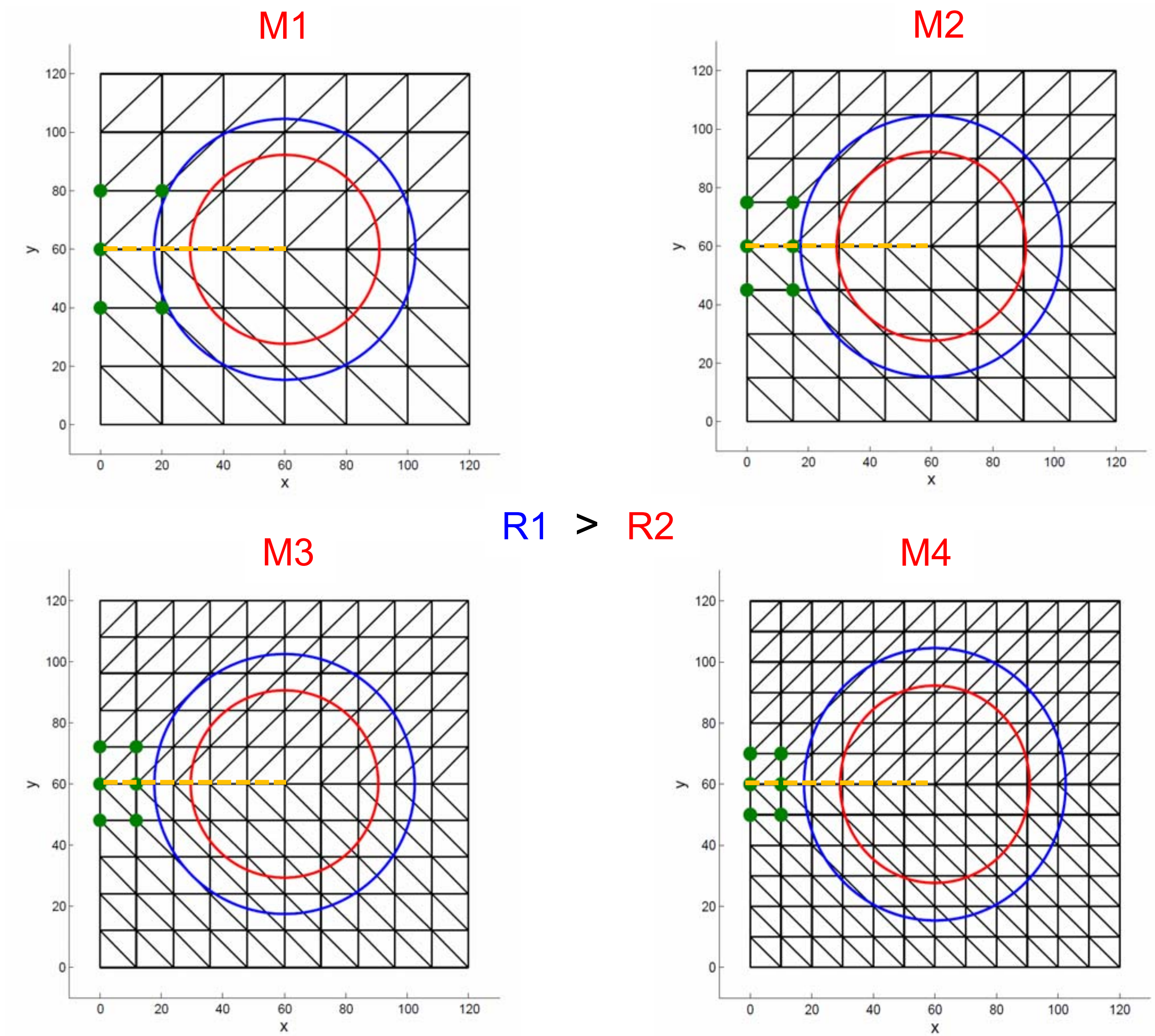
Configurational forces method

Quality assessment through local measures

Smoothness, enrichments and conditioning

Some improvements beyond...

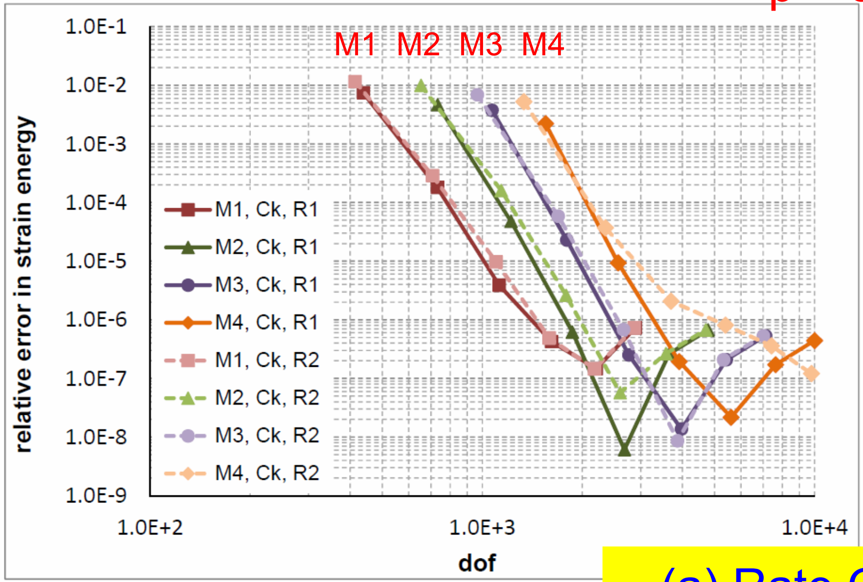
Geometric enrichment pattern



Convergence in terms of global values

p – convergence !

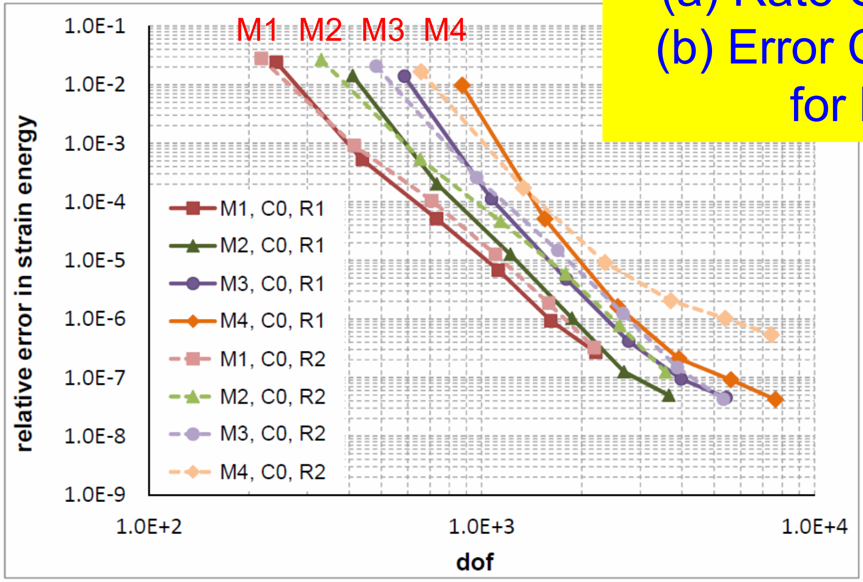
GFEM-C^k



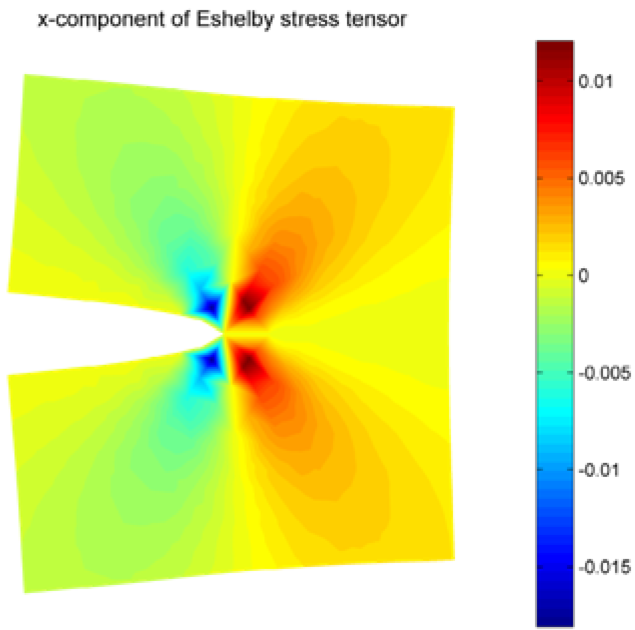
Geometric enrichment

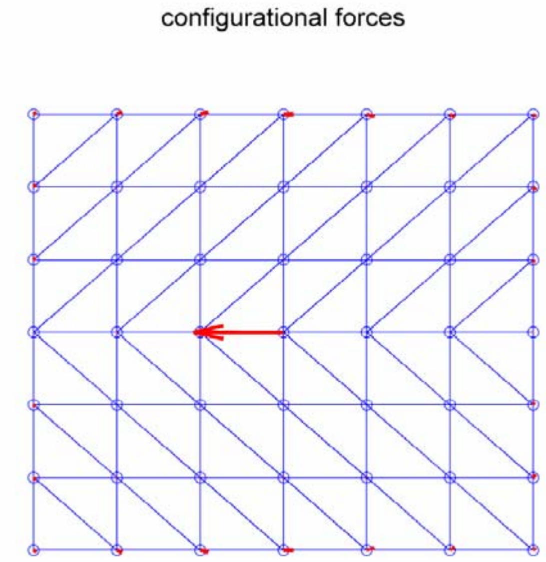
(a) Rate Ck > rate C0;
(b) Error Ck < error C0 for b=1,2,3,4.

GFEM-C^0



Local measure using configurational forces





Glaser and Steinmann, ***On material forces within the extended finite element method***. Proceedings of the sixth European Solid Mechanics Conference (2006)

Häusler, Lindhorst and Horst, ***Combination of the material force concept and the extended finite element method for mixed mode crack growth simulations***. International Journal for Numerical Methods in Engineering, 85 (2011)

Variational balance of material linear momentum

$$\boldsymbol{\Sigma}(\boldsymbol{u}) = \mathfrak{W}(\boldsymbol{u})\ \underline{\mathbb{I}} - \underline{\mathbb{L}}^T(\boldsymbol{u})\,\boldsymbol{\sigma}(\boldsymbol{u})$$

Eshelby tensor

$$\boldsymbol{\Sigma} = \{\Sigma_x, \Sigma_y, \Sigma_{xy}, \Sigma_{yx}\}^T \qquad \mathfrak{W} = \frac{1}{2}\,\boldsymbol{\sigma}^T \boldsymbol{\varepsilon}$$

$$\int_\Omega (\mathbb{L}\boldsymbol{v})^T \boldsymbol{\Sigma}\, l_z\, d\Omega = \int_\Omega (\boldsymbol{v})^T \boldsymbol{\varrho}\, l_z\, d\Omega$$

Inhomogeneity force

$$\boldsymbol{\varrho} = \{\varrho_x, \varrho_y\}^T \qquad \boldsymbol{v} = \{v_x, v_y\}^T$$

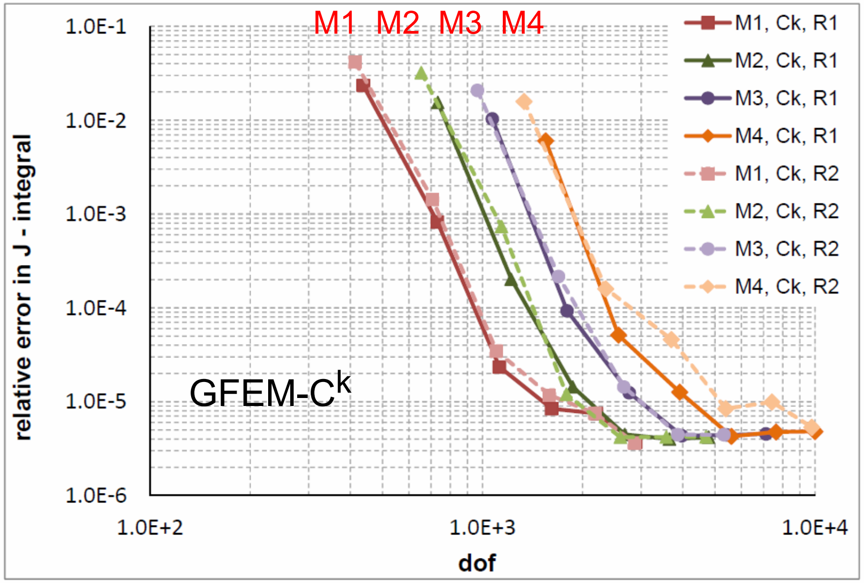
$$\underline{\mathbb{L}}(\boldsymbol{u}) = \underline{\mathbf{L}}\ \underline{\mathbf{I}}\ \boldsymbol{u} \qquad \underline{\mathbb{I}}^T = \{1,1,0,0\}$$

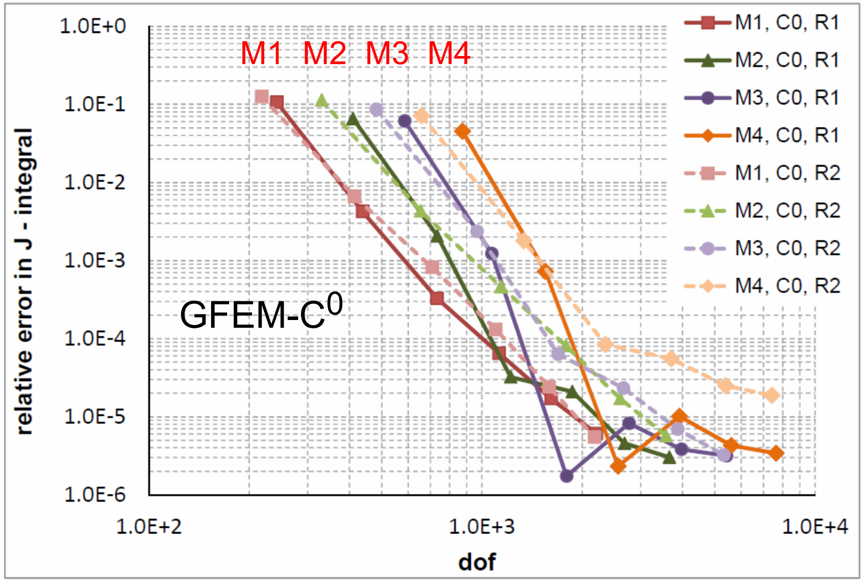
$$\mathbb{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{\mathbf{L}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{\mathbf{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Convergence in J-integral

Geometric enrichment

(a) Rate Ck > rate C0;
(b) Error Ck < error C0 for b=1,2,3,4.





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Continuous partition of unity with C^k-GFEM

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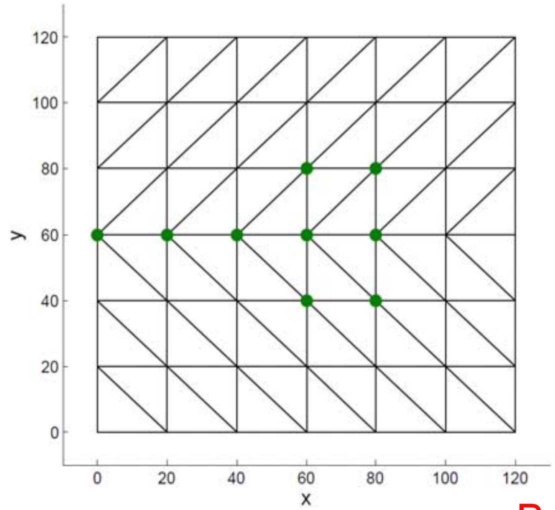
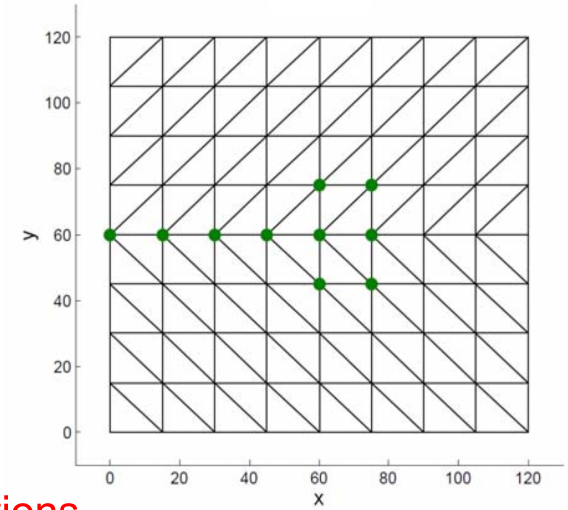
Some improvements beyond...

Topologic enrichment pattern

M1

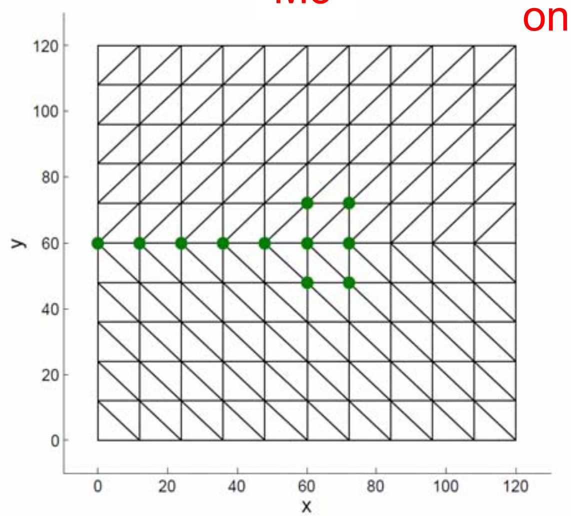


M2

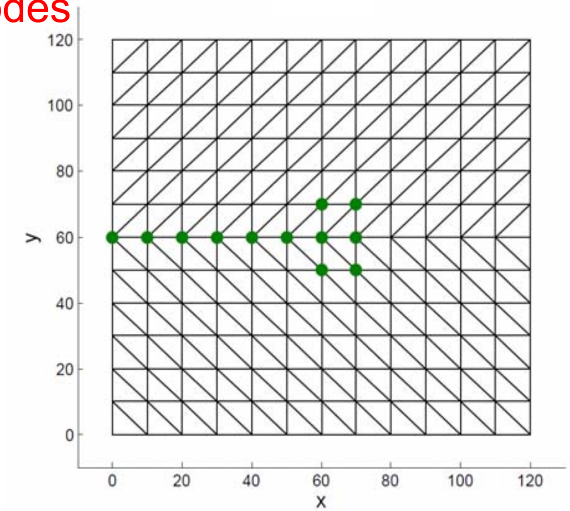


Branch functions and *p*-enrichment on green nodes

M3



M4



Convergence in terms of global and local values

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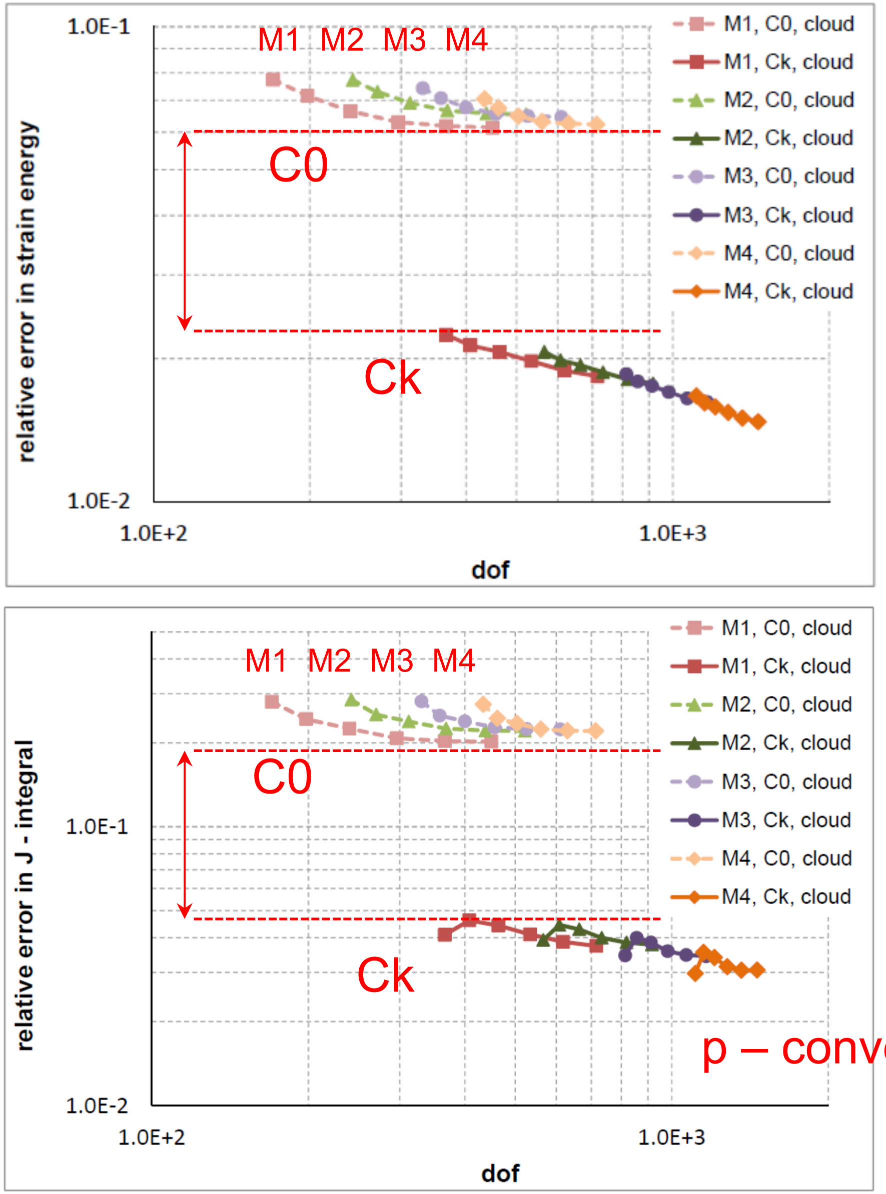
Energy

Topologic enrichment

J-integral



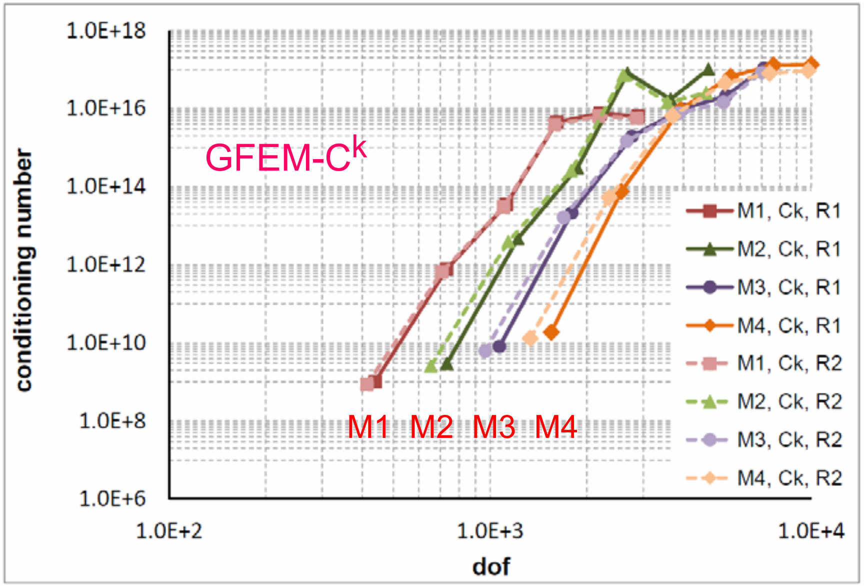
p – convergence !

Convergence rates for *h*-refinement

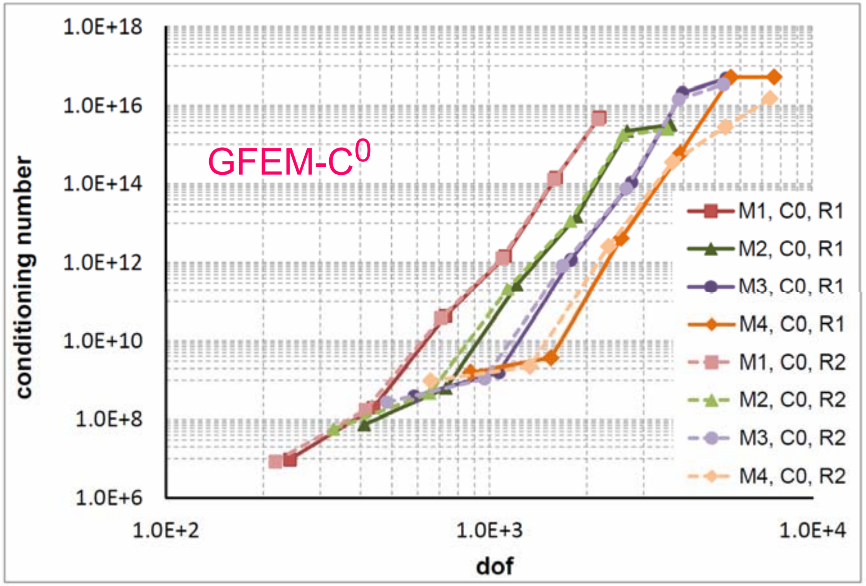
enrichment pattern	PoU	$b = 1$	$b = 2$	$b = 3$
geometric R1	$C^0(\Omega)$	0, 33	0, 91	1, 42
	$C^\infty(\Omega)$	0, 33	0, 91	1, 43
geometric R2	$C^0(\Omega)$	0, 24	0, 74	1, 07
	$C^\infty(\Omega)$	0, 34	0, 90	0, 88
topologic	$C^0(\Omega)$	0, 05	0, 04	0, 02
	$C^\infty(\Omega)$	0, 29	0, 26	0, 28
theoretical category A		0, 50	1, 00	1, 50
theoretical category B		0, 25	0, 25	0, 25

Condition number Nc

Geometric enrichment



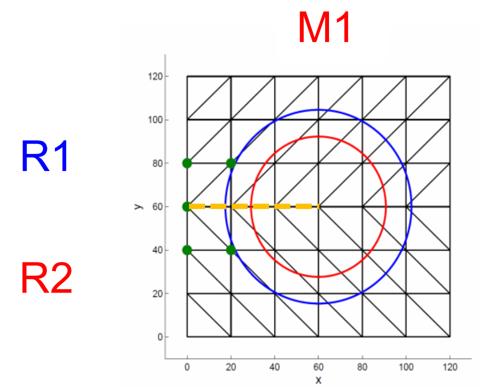
Nc C^k > Nc C^0

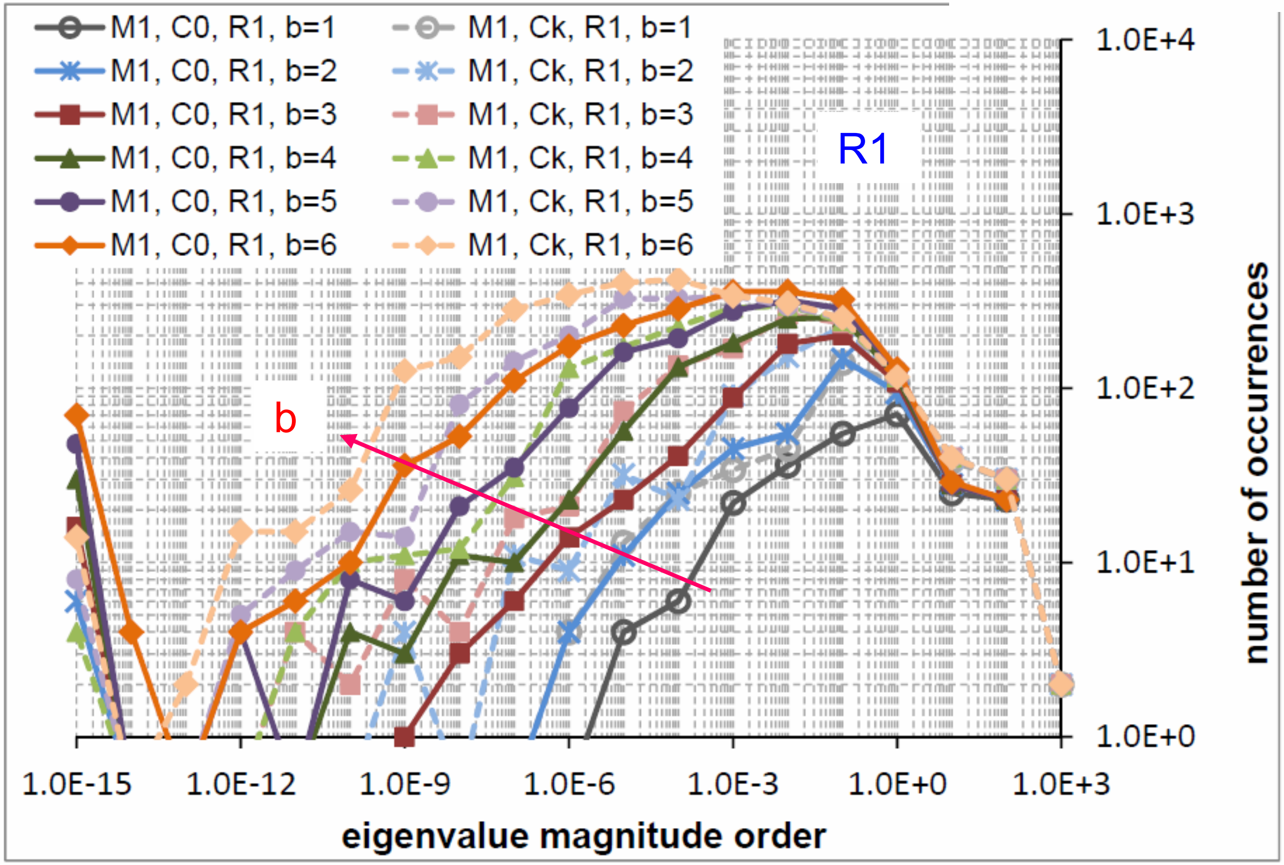


$$\text{cond. number} = \frac{\text{larger eigenvalue}}{\text{smaller eigenvelue} \neq 0}$$

Eigenvalues distribution X enrichment

- Geometric patern;
- Uniform p-enrichment





Eigenvalues distribution X enrichment

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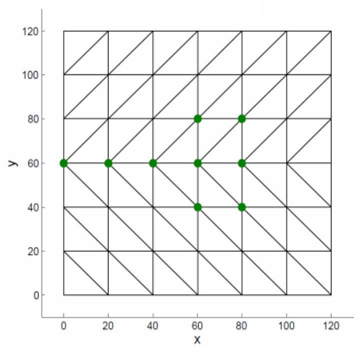

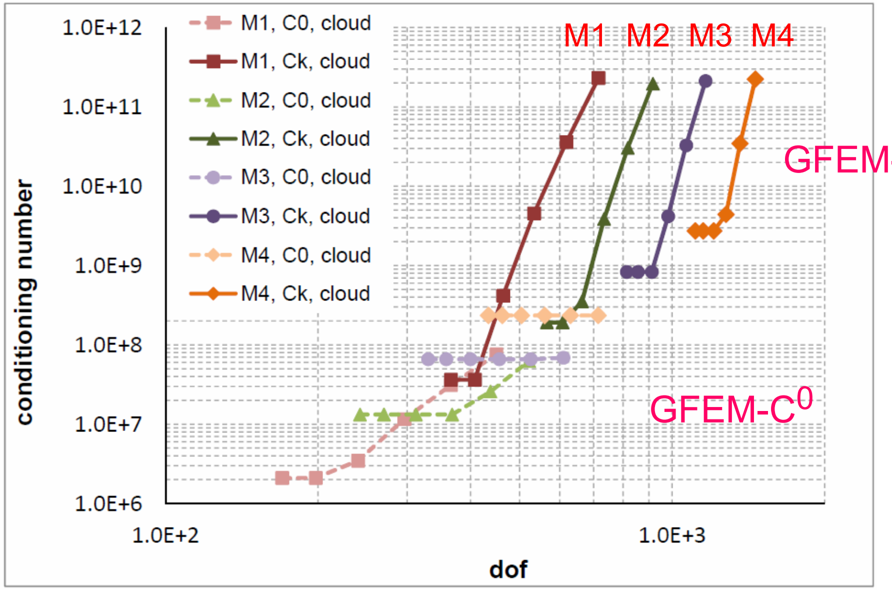

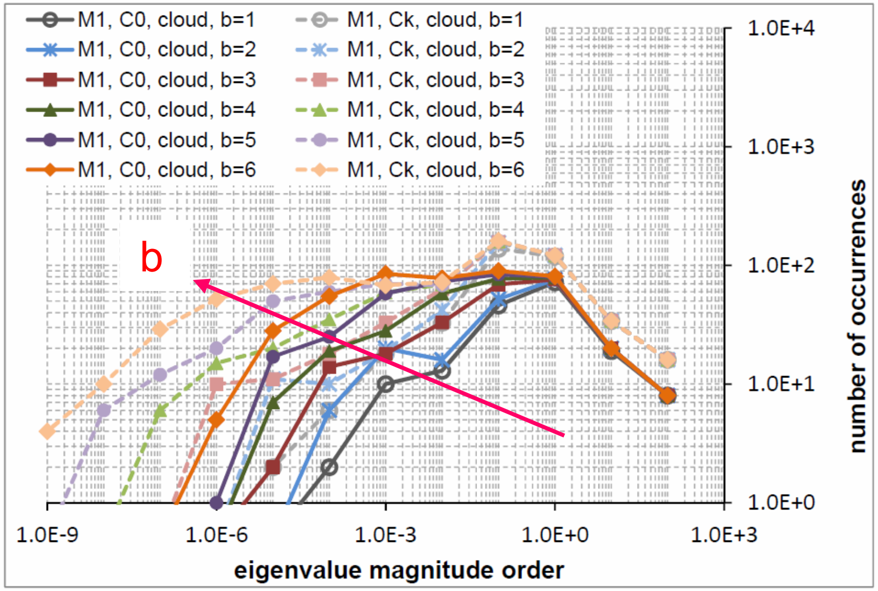

Exact error dispersion $(u_y - u_{yh})$

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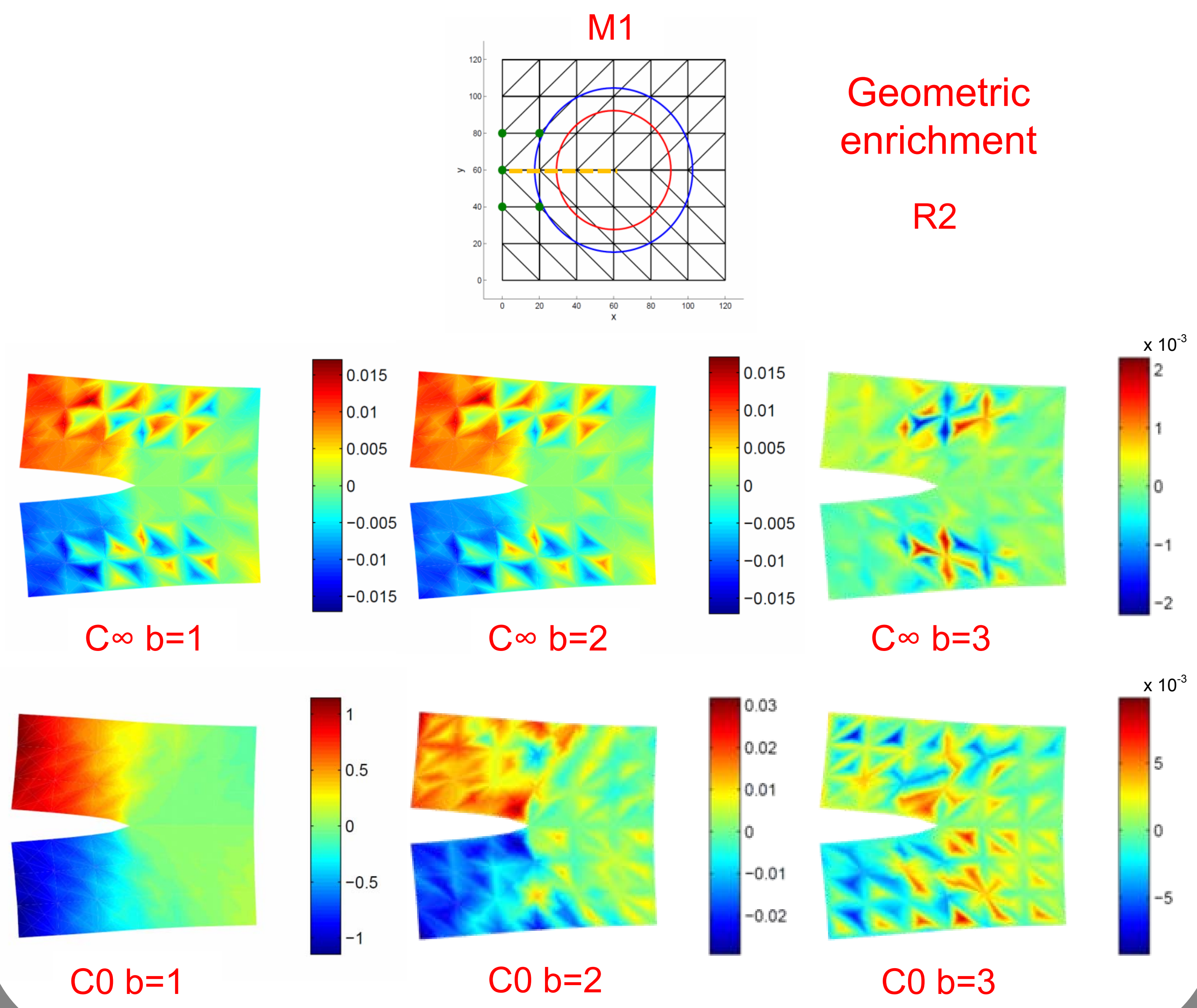

Concluding remarks

- Continuous stress fields around singularity provide better severity crack parameters

- Polynomial enrichments together with branch functions may adaptively improve the stress fields

- Continuity may conduct to better computation of nodal Eshelby forces

Acknowledgements

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Thank you!

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