ELSEVIER
Comput. Methods Appl. Mech. Engrg. 166 (1998) 363–378
Computer methods in applied mechanics and engineering

The general fundamental solution of the sixth-order Reissner and Mindlin plate bending models revisited

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Received 30 October 1997

Abstract

The sixth-order differential equation system of the Reissner and Mindlin plate bending models describe mathematically the plate problem, where lines normal to the mid-plane before deformation remain straight and inextensible in the deformed configuration, but not necessarily normal to the reference plane anymore. The non-normality condition is due to the consideration of transverse shearing strains, disregarded in the classical bi-harmonic Kirchhoff plate model. As the plate thickness is reduced and/or the transverse shear modulus is increased, the deformed configuration is less dependent of the transverse shear strains and, in the limit as the plate thickness approaches zero and/or the transverse shear modulus approaches ∞, the transverse shear strain effects vanish and the problem is exactly that described by the classical plate model. In this paper, we investigate the fundamental solutions of both the fourth-order Kirchhoff and the sixth-order Reissner and Mindlin plate models. We consider a transversely isotropic material and show that the fundamental solution of the bi-harmonic problem can be obtained directly from the general fundamental solution of the sixth-order plate problem, in the limit as the plate thickness approaches zero and/or the transverse shear modulus approaches ∞. This solution is in agreement with the analytical solution of an infinite thin clamped circular plate submitted to a unitary concentrated load acting at its center.

1. Introduction

The sixth-order Reissner [20–22] and Mindlin [18] plate models, accounting for the transverse shear strains disregarded in the classical bi-harmonic Kirchhoff [14] plate model, require the imposition of three boundary conditions (because the governing system is of the sixth order), solving in such way the Poisson/Kirchhoff paradox on a free edge [30,25]. These refined models are adequate for solving transverse shear strain sensitive moderately thick plate problems, where the transverse shearing strains contribute with significant effects in the plate behavior. For such problems, the use of the classical model is inappropriate for a satisfactory solution.

The sixth-order plate problem, whose natural formulation arises in differential form, was formulated in an integral form for the Reissner model by van der Weeën [32–34]. An integral formulation for the Mindlin model was developed by de Barcellos and Silva [8], whereas Westphal Jr. and de Barcellos [36] presented a unified integral formulation for both models. Constanda [5] presented a rigorous mathematical analysis of the sixth-order plate models by means of boundary integral formulations.

The well-known fundamental solution determined by van der Weeën [33,34] and since then extensively used, is however, not a general one. Westphal Jr., de Barcellos and Tomás Pereira [37] showed that an extended solution can be considered, the general fundamental solution, whose derived tensors in final form are simpler and potentially better than those of van der Weeën.

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0045-7825/98/$19.00
PII: S0045-7825(98)00101-7

In this paper, we show that the above cited general fundamental solution presents two important particularities: (a) in the limit as the plate thickness approaches zero the transversal displacement fundamental tensor component is compared with the corresponding general fundamental solution of the bi-harmonic plate model [37]. This process leads to a solution that is in complete accordance with that of an infinite thin clamped circular plate submitted to a centrally located concentrated unitary transverse force; and (b) the same previous conclusion is drawn if we consider a transversely isotropic material with a very high transverse shear modulus. The thin plate fundamental solution so obtained is of the same form as that considered by Costa Jr. and Brebbia [6].

The compatibility process between the general sixth-order Reissner and Mindlin and the general fourth-order Kirchhoff plate fundamental solutions lead us to the possibility of specifying the *best* set of values for the otherwise free general fundamental solution coefficients.

The indicial notation will be considered here, where Greek subscripts vary in the range 1 to 2 and Latin subscripts in the range 1 to 3, with repeated indices being summed according to Einstein's rule. Partial derivatives are expressed with the corresponding subscript preceded by a comma.

2. The basic abstract differential problem

Starting from the Lamé/Navier system for a given problem [37],

$$L_{ij}(\partial_Q)u_j(Q) = -F_{ij}(\partial_Q)q_j(Q)\,, \tag{1}$$

we write in the following the Lamé/Navier system for the auxiliary problem, which leads to its corresponding fundamental solution. In the above equation $L_{ij}(\partial_Q)$ is a given linear elliptic differential operator system with constant coefficients, $F_{ij}(\partial_Q)$ is a given differential operator system, ∂_Q being a symbol to indicate that the differential operators are applied in the field point Q, $q_j(Q)$ is a given vector, and $u_j(Q)$ are the generalized basic variables of the problem, the plate displacements.

The generalized displacements of the auxiliary problem are due to three generalized concentrated forces, such that [3]

$$u_j^*(Q) := U_{kj}(P, Q)e_k(P)\,, \tag{2}$$

where $u_j^*(Q)$ are the fundamental solution generalized displacements, $e_k(P)$ are the generalized unit-concentrated forces in the direction k and acting at the load point P, and $U_{kj}(P, Q)$ are the displacements in the direction j at the point Q due to the loadings $e_k(P)$.

Substituting Eq. (2) into Eq. (1) and observing the definition of the fundamental solution,

$$F_{ij}(\partial_Q)q_j(Q) := \delta(P, Q)e_i(P)\,, \tag{3}$$

being $\delta(P, Q)$ a Dirac's distribution at the point P, we obtain [33]

$$L_{ij}(\partial_Q)U_{kj}(P, Q) = -\delta(P, Q)\delta_{ik}\,, \tag{4}$$

which are the Lamé/Navier equations of the auxiliary problem, where δ_{ik} is the Kronecker's delta symbol. This system of differential operators can be further reduced to a simple differential equation, whose solution can be easily determined. For such a purpose, we apply the Hörmander's method [12]. Observing that

$$L_{ij}^{co}L_{kj} = L_{ij}L_{kj}^{co} = \det(\boldsymbol{L})\delta_{ik}\,, \tag{5}$$

being $\boldsymbol{L}^{co}$ the cofactor matrix of $\boldsymbol{L}$, and defining

$$U_{kj}(P, Q) := L_{kj}^{co}(\partial_Q)G(P, Q)\,, \tag{6}$$

we can write the system (4) as

$$\det(\boldsymbol{L})G(P, Q) = -\delta(P, Q)\,. \tag{7}$$

After solving the problem (7), the scalar fundamental solution $G(P, Q)$ is substituted into the system (6). In this way, the fundamental solution $U_{kj}(P, Q)$ of the Lamé/Navier system equations (4) is obtained.

This procedure was applied by Westphal Jr. et al. [37] to some well-known elliptic differential problems of the applied mechanics, including the problem under consideration here. In the following, we apply the method to the sixth-order plate problem, with the general scalar fundamental solution $G(P, Q)$ being determined in a slightly different manner.

3. The sixth-order plate model general fundamental solution

Consider a plate of uniform thickness $h \equiv 2c > 0$, homogeneous and transversely isotropic, referred to a three-dimensional Cartesian coordinate system, with the thickness axis x_3 normal to the plane of isotropy, and with the $x_1 - x_2$ reference plane lying on the plate mid-surface. Following the notation of Jones [13], the material constants are: $E_1 \equiv E$ and $\nu_{12} \equiv \nu$, the isotropic in-plane elasticity modulus and Poisson coefficient, respectively, and E_3, ν_{13}, ν_{31} and G_3, the transverse elasticity modulus, transverse Poisson coefficients and transverse shear modulus, respectively. The relation $E_1\nu_{31} = E_3\nu_{13}$ should be observed, leading to the five constants which characterize such a material. Further, the in-plane shear modulus meets the relation $G = E/(2(1+\nu))$ and we define

$$k_E := \sqrt{\frac{E_3}{E}}, \qquad k_G := \frac{G_3}{G}, \qquad k_\nu := \frac{\nu_3}{\nu}, \tag{8a}$$

where

$$\nu_3 := \sqrt{\nu_{13}\nu_{31}}\,. \tag{8b}$$

The three constants k_E, k_G and k_ν, characterize the material transverse isotropy, being the material isotropic if $k_E = k_G = k_\nu := 1$. The stress and strain tensors at an arbitrary point $\boldsymbol{x} := (x_1, x_2, x_3)$ of a three-dimensional Cartesian coordinate system are denoted by $\sigma_{ij}(\boldsymbol{x})$ and $\varepsilon_{ij}(\boldsymbol{x})$, respectively. In the following, the five material constants are specified by E, ν, k_E, k_G and k_ν, and the constitutive relations for the linear three-dimensional theory of elasticity are expressed by

$$\varepsilon_{\alpha\beta}(\boldsymbol{x}) = \frac{1}{E}\left[(1+\nu)\sigma_{\alpha\beta}(\boldsymbol{x}) - \nu\left(\sigma_{\gamma\gamma}(\boldsymbol{x}) + \frac{k_\nu}{k_E}\sigma_{33}(\boldsymbol{x})\right)\delta_{\alpha\beta}\right], \tag{9a}$$

$$\varepsilon_{\alpha 3}(\boldsymbol{x}) = \frac{1+\nu}{k_G E}\sigma_{\alpha 3}(\boldsymbol{x}), \tag{9b}$$

$$\varepsilon_{33}(\boldsymbol{x}) = \frac{1}{k_E^2 E}[\sigma_{33}(\boldsymbol{x}) - \nu k_\nu k_E \sigma_{\gamma\gamma}(\boldsymbol{x})], \tag{9c}$$

or

$$\sigma_{\alpha\beta}(\boldsymbol{x}) = G\left[\varepsilon_{\alpha\beta}(\boldsymbol{x}) + \varepsilon_{\beta\alpha}(\boldsymbol{x}) + \frac{2\nu}{\beta}((1 + k_\nu^2\nu)\varepsilon_{\gamma\gamma}(\boldsymbol{x}) + (1+\nu)k_\nu k_E \varepsilon_{33}(\boldsymbol{x}))\delta_{\alpha\beta}\right], \tag{10a}$$

$$\sigma_{\alpha 3}(\boldsymbol{x}) = \frac{k_G E}{1+\nu}\varepsilon_{\alpha 3}(\boldsymbol{x}), \tag{10b}$$

$$\sigma_{33}(\boldsymbol{x}) = \frac{k_E E}{\beta}[k_\nu \nu \varepsilon_{\gamma\gamma}(\boldsymbol{x}) + (1-\nu)k_E \varepsilon_{33}(\boldsymbol{x})], \tag{10c}$$

with

$$\beta \equiv \beta(\nu, \nu_3) := 1 - \nu - 2\nu_3^2\,. \tag{10d}$$

The problem is investigated for the particular case of bending behavior, case (a) of Fig. 1. Considering that only a transverse loading $\sigma^{(1)}(\bar{\boldsymbol{x}})$ is applied on the plate faces ($A := \sigma^{(1)}$ and $B := 0$ in Fig. 1) it implies (Eq. (1))

$$\{\boldsymbol{q}(\bar{\boldsymbol{x}})\}_3 := \{0 \quad 0 \quad \sigma^{(1)}(\bar{\boldsymbol{x}})\}^{\mathrm{T}}\,. \tag{11}$$

We define the applied loading and the two-dimensional plate variables with a superscript 1 enclosed by

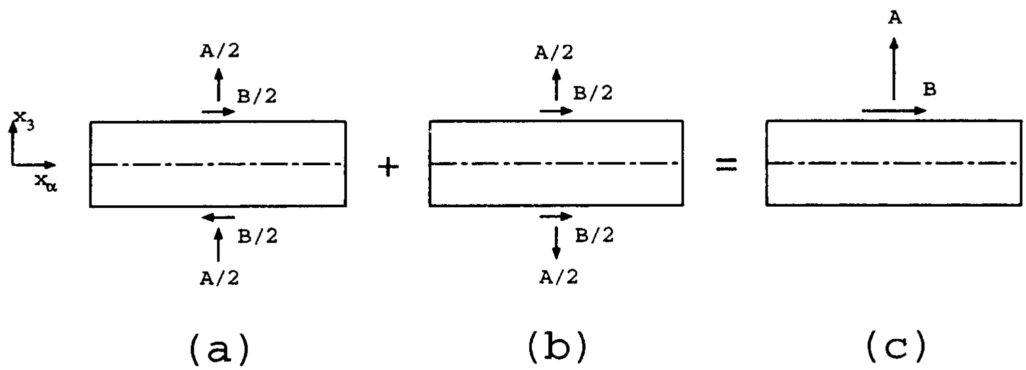

Fig. 1. A general problem (c) composed as the sum of a bending (a) and a stretching (b) problem.

parenthesis. This notation reveals to be very practical when used in connection with high-order plate models (see [38]).

Starting from the displacement assumption [15]

$$v_\alpha(\boldsymbol{x}) := \phi_\alpha^{(1)}(\bar{\boldsymbol{x}}) x_3 , \tag{12a}$$

$$v_3(\boldsymbol{x}) := \phi_3^{(1)}(\bar{\boldsymbol{x}}) , \tag{12b}$$

we obtain for the stress distribution along the plate thickness [23]

$$\sigma_{\alpha\beta}(\boldsymbol{x}) = \frac{3}{2c^2}\frac{x_3}{c}\,\sigma_{\alpha\beta}^{(1)}(\bar{\boldsymbol{x}}) , \tag{13a}$$

$$\sigma_{\alpha 3}(\boldsymbol{x}) = \frac{3}{4c}\left(1 - \frac{x_3^2}{c^2}\right)\sigma_\alpha^{(1)}(\bar{\boldsymbol{x}}) , \tag{13b}$$

$$\sigma_{33}(\boldsymbol{x}) = \frac{1}{4}\left(3\frac{x_3}{c} - \frac{x_3^3}{c^3}\right)\sigma^{(1)}(\bar{\boldsymbol{x}}) , \tag{13c}$$

together with the plate equilibrium equations

$$\sigma_{\alpha\beta,\beta}^{(1)}(\bar{\boldsymbol{x}}) - \sigma_\alpha^{(1)}(\bar{\boldsymbol{x}}) = 0 , \tag{14a}$$

$$\sigma_{\alpha,\alpha}^{(1)}(\bar{\boldsymbol{x}}) + \sigma^{(1)}(\bar{\boldsymbol{x}}) = 0 . \tag{14b}$$

In the above equations $\bar{\boldsymbol{x}} := (x_1, x_2)$ is a point on the reference plane of the plate (or on a plane parallel to the reference plane), $v_i(\boldsymbol{x})$ are the displacement components, $\phi_i^{(1)}(\bar{\boldsymbol{x}})$ are the plate displacement components, and the plate stresses are defined as

$$\sigma_{\alpha\beta}^{(1)}(\bar{\boldsymbol{x}}) := \int_{-c}^{c} \sigma_{\alpha\beta}(\boldsymbol{x}) x_3 \, \mathrm{d}x_3 , \tag{15a}$$

$$\sigma_\alpha^{(1)}(\bar{\boldsymbol{x}}) := \int_{-c}^{c} \sigma_{\alpha 3}(\boldsymbol{x}) \, \mathrm{d}x_3 . \tag{15b}$$

We define a 3×3 matrix $\boldsymbol{\tau}^{(1)}(\bar{\boldsymbol{x}})$, whose elements are the plate stresses and the plate loading, such that

$$[\boldsymbol{\tau}^{(1)}(\bar{\boldsymbol{x}})]_{3\times 3} := \begin{bmatrix} \sigma_{11}^{(1)}(\bar{\boldsymbol{x}}) & \sigma_{12}^{(1)}(\bar{\boldsymbol{x}}) & \sigma_1^{(1)}(\bar{\boldsymbol{x}}) \\ & \sigma_{22}^{(1)}(\bar{\boldsymbol{x}}) & \sigma_2^{(1)}(\bar{\boldsymbol{x}}) \\ & & \sigma^{(1)}(\bar{\boldsymbol{x}}) \end{bmatrix}_{\mathrm{Sym}} , \tag{16}$$

where the matrix subscript Sym is used for symmetric matrices. Note that these are all the two-dimensional variables in the RHS of Eq. (13).

The plate loading $\sigma^{(1)}(\bar{\boldsymbol{x}})$ (an a priori non integrated and known *plate variable*) is defined as the element

$\tau_{33}^{(1)}(\bar{x})$ in order to complete the above defined plate stress tensor. The actual plate model corresponds to a transversely inextensible one, as the plate transverse displacement given in Eq. (12b) is constant along the plate thickness (see however Eq. (20) and its subsequent discussion). One can note that there is no one plate stress associated to $\sigma_{33}(x)$ in Eqs. (15). For plate models of high order, which take transverse direct deformation into account, integrated components of $\sigma_{33}(x)$ take the place of the transverse loading in the respective high-order tensors. For the two terms (third-order polynomial) displacement approximation of the twelfth-order (differential equation system) Reissner plate model [15], it results [38]

$$[\tau^{(2)}(\bar{x})]_{3\times 3} := \begin{bmatrix} \sigma_{11}^{(2)}(\bar{x}) & \sigma_{12}^{(2)}(\bar{x}) & \sigma_1^{(2)}(\bar{x}) \\ & \sigma_{22}^{(2)}(\bar{x}) & \sigma_2^{(2)}(\bar{x}) \\ & & \sigma^{(2)}(\bar{x}) \end{bmatrix}_{\mathrm{Sym}} \equiv \begin{bmatrix} P_{11}(\bar{x}) & P_{12}(\bar{x}) & S_1(\bar{x}) \\ & P_{22}(\bar{x}) & S_2(\bar{x}) \\ & & T(\bar{x}) \end{bmatrix}_{\mathrm{Sym}}, \tag{17}$$

where we presented the matrix elements in the RHS with the notation of Reissner [24]. $\sigma_2^{(2)}(\bar{x}) \equiv T(\bar{x})$ accounts for a plate stress obtained directly from $\sigma_{33}(x)$ (see [38]).

Applying the Hellinger–Reissner variational principle [23,35] we obtain for Eq. (1), together with Eq. (11),

$$[L]_{3\times 3} := D\frac{1-\nu}{2}\begin{bmatrix} \Delta - \lambda^2 + \hat{\nu}\,\partial_{11}^2 & \hat{\nu}\,\partial_{12}^2 & -\lambda^2\,\partial_1 \\ \hat{\nu}\,\partial_{12}^2 & \Delta - \lambda^2 + \hat{\nu}\,\partial_{22}^2 & -\lambda^2\,\partial_2 \\ \lambda^2\,\partial_1 & \lambda^2\,\partial_2 & \lambda^2\Delta \end{bmatrix}, \tag{18a}$$

$$\{u(\bar{x})\}_3 := \{\varphi_1^{(1)}(\bar{x}) \quad \varphi_2^{(1)}(\bar{x}) \quad \varphi_3^{(1)}(\bar{x})\}^{\mathrm{T}}, \tag{18b}$$

$$[F]_{3\times 3} := \begin{bmatrix} \dfrac{\nu_3 k_G}{(1-\nu)\lambda^2 k_E}\,\partial_1 & 0 & 0 \\ & \dfrac{\nu_3 k_G}{(1-\nu)\lambda^2 k_E}\,\partial_2 & 0 \\ & & 1 \end{bmatrix}_{\mathrm{Sym}}, \tag{18c}$$

where $\partial_\alpha \equiv \partial/\partial x_\alpha$ and $\partial_{\alpha\beta}^2 \equiv \partial^2/(\partial x_\alpha\,\partial x_\beta)$ for derivatives in matrix notation, $\Delta := \partial_{\alpha\alpha}^2$ is the two-dimensional Laplace's operator, $D := 2Ec^3/(3(1-\nu^2))$ is the plate flexural rigidity, $\hat{\nu} := (1+\nu)/(1-\nu)$, and

$$\lambda^2 := \frac{3k^2 k_G}{c^2}, \tag{19}$$

with $k^2 := 5/6$ for the Reissner plate model, whereas for the Mindlin plate model $k^2 := \pi^2/12$ or $4\sqrt{(1-(1-2\nu)k^2/(2(1-\nu)))(1-k^2)} := (2-k^2)^2$ [18].[3]

The plate displacements obtained from the Hellinger–Reissner principle are [10]

$$\varphi_\alpha^{(1)}(\bar{x}) := \frac{3}{2c^2}\int_{-c}^{c} v_\alpha(x)\,\frac{x_3}{c}\,\mathrm{d}x_3\,, \tag{20a}$$

$$\varphi_3^{(1)}(\bar{x}) := \frac{3}{4c}\int_{-c}^{c} v_3(x)\left(1-\frac{x_3^2}{c^2}\right)\mathrm{d}x_3\,, \tag{20b}$$

where $v_i(x)$ are the three-dimensional displacements of a point of the plate. If they are exactly of the form given in Eq. (12) then $\varphi_i^{(1)}(\bar{x}) = \phi_i^{(1)}(\bar{x})$, showing that the displacements represented in Eq. (20) (Reissner) are a generalization of those in Eq. (12) (Mindlin).

Remark that in the differential governing system (18a) the only transverse material parameter present is k_G. In the loading operator F_{ij}, Eq. (18c), the additional parameter ν_3/k_E comes into play, but not $1/E_3$ which would be associated with σ_{33} or ε_{33}, Eqs. (9c) and (10c), respectively.

[3] The correction factors of Mindlin are valid for isotropic materials. For non-isotropic materials analogous constants can be obtained for the appropriate medium (see [18]).

Hereafter, we work with the system of two-dimensional variables and do not show the explicit dependence on $\bar{x}$ anymore.

As shown by van der Weeën [33], Eq. (7) reads

$$D^3\left(\frac{1-\nu}{2}\right)^2 \lambda^2 \Delta^2(\Delta - \lambda^2)G(P,Q) = -\delta(P,Q)\,, \tag{21}$$

with the transversal shear coefficient k_G included in the parameter λ^2, Eq. (19).

To determine the general solution $U_{ij}(P,Q)$ we solve the homogeneous problem related to Eq. (21), namely

$$\Delta^2(\Delta - \lambda^2)V(r) = 0\,, \quad r \neq 0 \tag{22}$$

where $r := |P - Q|$ is the distance from P to Q. Eq. (21) is invariant under rotations, so its solution depends only on the radial variable r [4]. Such solution can be written in the form

$$V(r) := V_1(r) + V_2(r)\,, \tag{23a}$$

where

$$\Delta^2 V_1(r) = 0\,, \tag{23b}$$

$$(\Delta - \lambda^2)V_2(r) = 0\,. \tag{23c}$$

Eq. (23c) can be expressed as

$$z^2 V_{2,zz} + zV_{2,z} - z^2 V_2 = 0\,, \tag{24}$$

where $z := \lambda r$.

The general solutions of Eqs. (23b) and (24) are

$$V_1(r) = B_1 + B_2 \ln r + B_3 r^2 + B_4 r^2 \ln r\,, \tag{25a}$$

$$V_2(z) = B_5 K_0(z) + B_6 I_0(z)\,, \tag{25b}$$

where $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of first and second kind, respectively [1], and B_1 to B_6 are constants.

We express the general solution of Eq. (22) as

$$G(z) = C_1 K_0(z) + C_2 r^2 \ln r + C_3 \ln r + C_4 r^2 + C_5 + C_6 I_0(z)\,. \tag{26}$$

This solution was expressed by van der Weeën [33], Silva [28], Westphal Jr. et al. [37] in the form

$$G(z) = D_1 K_0(z) + D_2 z^2 \ln z + D_3 \ln z + D_4 z^2 + D_5 + D_6 I_0(z)\,, \tag{27}$$

with the relations between the coefficients in Eqs. (26) and (27) being

$$\begin{aligned} &C_1 := D_1\,, \quad C_2 := D_2\lambda^2\,, \quad C_3 := D_3 \\ &C_4 := (D_4 + D_2 \ln \lambda)\lambda^2\,, \quad C_5 := D_5 + D_3 \ln \lambda\,, \quad C_6 := D_6\,. \end{aligned} \tag{28}$$

In the following, we consider Eq. (26) in order to determine the constants C_1 to C_6 that meet Eq. (7). It is clear that Eq. (27) could equally well be used, but in applying Eq. (26) we do not have the coupling effect between the six linearly independent functions which build up the general solutions in Eq. (25) forming the fundamental solution, as is the case with the constants of Eq. (27) that comply with Eq. (28). We express the general scalar fundamental solution G as a function of r only, namely $G(r)$, since for each problem the plate thickness h and the transverse shear modulus factor k_G are fixed, implying the factor λ to be a constant.

The significance of our fundamental solution $G(r)$, Eq. (26), is its general representation, that is, the most general solution of the governing differential operator. The most frequently used solution of van der Weeën [33] can be directly determined from Eq. (27) if the free coefficients are set as $D_4 := -D_1/4$ and $D_5 := 0$ [28,36]. Constanda [5] considered Eq. (26) with $C_4 = C_5 := 0$. A judicious investigation of the operator L_{ij}^{co} of Eq. (6), that is

$$[\boldsymbol{L}^{co}]_{3\times 3} = \left[D\frac{1-\nu}{2}\right]^2$$

$$\begin{bmatrix} \lambda^2[(1+\hat{\nu})\Delta^2 - (\hat{\nu}\Delta + \lambda^2)\,\partial_{11}^2] & -\lambda^2[\hat{\nu}\Delta + \lambda^2]\,\partial_{12}^2 & -\lambda^2[\Delta - \lambda^2]\,\partial_1 \\ -\lambda^2[\hat{\nu}\Delta + \lambda^2]\,\partial_{12}^2 & \lambda^2[(1+\hat{\nu})\Delta^2 - (\hat{\nu}\Delta + \lambda^2)\,\partial_{22}^2] & -\lambda^2[\Delta - \lambda^2]\,\partial_2 \\ \lambda^2[\Delta - \lambda^2]\,\partial_1 & \lambda^2[\Delta - \lambda^2]\,\partial_2 & [\Delta - \lambda^2][(1+\hat{\nu})\Delta - \lambda^2] \end{bmatrix}, \qquad (29)$$

reveals that the differential operator L_{33}^{co} involves a non-differential term, $[D\lambda^2(1-\nu)/2]^2$, that does not eliminate the constant C_5 and D_5 of $G(z)$, Eqs. (26) and (27), respectively. Such fact seems to have been ignored by the BEM community, with the exception of the related works of the present authors. Clearly this does not imply that the other fundamental solutions are wrong. This question concerns a free coefficient function, while the essential ones which generate a Dirac's distribution are correctly posed.

Using the general solution equation (26), that meets Eq. (7), and the differential operator $\det(\boldsymbol{L})$ given in Eq. (21), we calculate the general fundamental solution $U_{ij}(P, Q)$ using Eq. (6).

We need now to investigate the solution of the problem depicted in Eq. (22) at the singular point $r = 0$, when Eq. (21) must then be considered. This is a generalized function and the analysis should be performed according to the theory of distributions[4]

$$\langle G(r), \det(\boldsymbol{L})\phi(r)\rangle = \langle \det(\boldsymbol{L})G(r), \phi(r)\rangle\,, \quad \phi(r) \in \mathscr{D}(\mathbb{R}^2)\,, \qquad (30)$$

where $\langle a, b\rangle$ denotes the duality pairing of $a \in \mathscr{D}'(\mathbb{R}^2)$ and $b \in \mathscr{D}(\mathbb{R}^2)$, being $\mathscr{D}(\mathbb{R}^2)$ the space of test functions in $\mathbb{R}^2$ and $\mathscr{D}'(\mathbb{R}^2)$ its dual space, the space of distributions. From Eq. (7) it follows [28,37]

$$\langle G(r), \det(\boldsymbol{L})\phi(r)\rangle = -\phi(0)\,. \qquad (31)$$

Let $S(P, \varepsilon)$ be a sphere with boundary Γ_ε and radius ε centered at the singular point P. Next exclude the associated region Ω_ε from the infinite one Ω^*, integrate by parts, and take the limit $\varepsilon \to 0$. This results [28]

$$\lim_{\varepsilon\to 0}\Bigg\{ \overbrace{-\int_{\Gamma_\varepsilon} G\Delta(\Delta - \lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}\Gamma}^{1} + \overbrace{\int_{\Gamma_\varepsilon} \frac{\mathrm{d}G}{\mathrm{d}r}\Delta(\Delta - \lambda^2)\phi\,\mathrm{d}\Gamma}^{2} - \overbrace{\int_{\Gamma_\varepsilon} \Delta G\,(\Delta - \lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}\Gamma}^{3}$$
$$+ \overbrace{\int_{\Gamma_\varepsilon} \Delta\frac{\mathrm{d}G}{\mathrm{d}r}(\Delta - \lambda^2)\phi\,\mathrm{d}\Gamma}^{4} - \overbrace{\int_{\Gamma_\varepsilon} \Delta^2 G\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}\Gamma}^{5} + \overbrace{\int_{\Gamma_\varepsilon} \Delta^2\frac{\mathrm{d}G}{\mathrm{d}r}\phi\,\mathrm{d}\Gamma}^{6}$$
$$+ \overbrace{\int_{\Omega^* - \Omega_\varepsilon} \phi\Delta^2(\Delta - \lambda^2)G\,\mathrm{d}\Omega}^{7}\Bigg\} = -\frac{4}{D^3(1-\nu)^2\lambda^2}\,\phi(0)\,. \qquad (32)$$

The derivatives of G, Eq. (26), are

$$\frac{\mathrm{d}G}{\mathrm{d}r} = -\lambda C_1 K_1(\lambda r) + C_2(2\ln r + 1)r + C_3\frac{1}{r} + 2C_4 r + \lambda C_6 I_1(\lambda r)\,, \qquad (33a)$$

$$\Delta G = \lambda^2 C_1 K_0(\lambda r) + 4C_2(\ln r + 1) + 4C_4 + \lambda^2 C_6 I_0(\lambda r)\,, \qquad (33b)$$

$$\Delta\frac{\mathrm{d}G}{\mathrm{d}r} = -\lambda^3 C_1 K_1(\lambda r) + 4C_2\frac{1}{r} + \lambda^3 C_6 I_1(\lambda r)\,, \qquad (33c)$$

$$\Delta^2 G = \lambda^4 C_1 K_0(\lambda r) + \lambda^4 C_6 I_0(\lambda r)\,, \qquad (33d)$$

[4] Using the multi-index notation [12,29,31]

$$\langle G(r), D^\alpha\phi(r)\rangle = (-1)^{|\alpha|}\langle {}^t D^\alpha G(r), \phi(r)\rangle\,,$$

and for the present case $D^\alpha \equiv \det(\boldsymbol{L})$; as D^α is a self-adjoint polynomial in Δ with constant coefficients and $|\alpha|$ is even, this implies that the transpose of D^α, ${}^t D^\alpha$, coincides with D^α.

$$\Delta^2 \frac{\mathrm{d}G}{\mathrm{d}r} = -\lambda^5 C_1 K_0(\lambda r) + \lambda^5 C_6 I_1(\lambda r)\,, \tag{33e}$$

$$\Delta^2(\Delta - \lambda^2)G = 0\,. \tag{33f}$$

We have on the sphere boundary $r = \varepsilon$ and $\mathrm{d}\Gamma = \varepsilon\,\mathrm{d}\theta$, and for small arguments ε

$$K_0(\lambda\varepsilon) = -\left(\ln \frac{\lambda\varepsilon}{2} + \gamma\right), \tag{34a}$$

$$K_1(\lambda\varepsilon) = \frac{1}{\lambda\varepsilon}, \tag{34b}$$

$$I_0(\lambda\varepsilon) = 0\,, \tag{34c}$$

$$I_1(\lambda\varepsilon) = \frac{\lambda\varepsilon}{2}, \tag{34d}$$

where γ is the Euler constant.

As ϕ is a test function

$$\left|\Delta(\Delta - \lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| \le M_1\,, \qquad |\Delta(\Delta - \lambda^2)\phi| \le M_2\,,$$
$$\left|(\Delta - \lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| \le M_3\,, \qquad |(\Delta - \lambda^2)\phi| \le M_4\,, \qquad \left|\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| \le M_5\,, \tag{35}$$

where M_1 to M_5 are bounded constants.

We can now analyse the several integrals in Eq. (32). The last boundary integral, the integral number 6, contributes for the singular behavior of the problem, the integrals number 1, 3 and 5 are cancelated and the remaining ones, the integrals 2 and 4, are non-trivial, and should consequently be zeroed. This regularization is performed through judicious choices of the coefficients C_1 to C_6. It can be verified that we should have

$$C_2 = \left(\frac{\lambda}{2}\right)^2 C_1\,, \tag{36a}$$

$$C_3 = C_1\,. \tag{36b}$$

Finally, from the integral number 6,

$$C_1 = \frac{2}{\pi D^3 (1 - \nu)^2 \lambda^6}\,. \tag{37}$$

The coefficient C_6 should be null, according to the regularity condition at infinity for the fundamental solution [3,4,36]. We do not have any imposition on the coefficients C_4 and C_5; they are free coefficients.

We define two constants F_4 and F_5 such that

$$C_4 := \lambda^2 F_4 C_1\,, \tag{38a}$$

$$C_5 := F_5 C_1\,. \tag{38b}$$

The system (6) give us

$$U_{\alpha\beta} = \frac{1}{8\pi D(1 - \nu)}\{[8B(z) - (1 - \nu)(2\ln r + 1 + 8F_4)]\delta_{\alpha\beta} - [8A(z) + 2(1 - \nu)]r_{,\alpha}r_{,\beta}\}\,,$$
$$U_{\alpha 3} = -U_{3\alpha} = \frac{1}{8\pi D}[2\ln r + 1 + 8F_4]rr_{,\alpha}\,,$$
$$U_{33} = \frac{1}{8\pi D\lambda^2}\left\{\lambda^2 r^2(\ln r + 4F_4) - \frac{8}{1 - \nu}\left[\ln r + \frac{3 - \nu}{2}(1 + 4F_4) - \frac{1 - \nu}{2}F_5\right]\right\}, \tag{39}$$

where it was defined [33]

$$A(z) := K_0(z) + \frac{2}{z}\left(K_1(z) - \frac{1}{z}\right), \tag{40a}$$

$$B(z) := K_0(z) + \frac{1}{z}\left(K_1(z) - \frac{1}{z}\right). \tag{40b}$$

The functions $A(z)$ and $B(z)$ have the following behavior for small arguments ε

$$A(\lambda\varepsilon) = -\frac{1}{2}, \tag{41a}$$

$$B(\lambda\varepsilon) = -\frac{1}{2}\left[\ln\left(\frac{\lambda\varepsilon}{2}\right) + \gamma + \frac{1}{2}\right], \tag{41b}$$

this being a good approximation for $\lambda\varepsilon < 7 \times 10^{-5}$.

Defining two variables α_1 and α_2

$$2 \ln \alpha_1 := 1 + 8F_4, \tag{42a}$$

$$2 \ln \alpha_2 := (3 - \nu)(1 + 4F_4) - (1 - \nu)F_5, \tag{42b}$$

we obtain

$$\begin{aligned} U_{\alpha\beta}(r) &= \frac{1}{4\pi D(1-\nu)}\{[4B(\lambda r) - (1-\nu)\ln(\alpha_1 r)]\delta_{\alpha\beta} - [4A(\lambda r) + (1-\nu)]r_{,\alpha}r_{,\beta}\}, \\ U_{\alpha 3}(r) &= -U_{3\alpha}(r) = \frac{1}{4\pi D}\ln(\alpha_1 r) r r_{,\alpha}, \\ U_{33}(r) &= \frac{1}{8\pi D}\left\{r^2\left[\ln(\alpha_1 r) - \frac{1}{2}\right] - \frac{8}{(1-\nu)\lambda^2}\ln(\alpha_2 r)\right\}. \end{aligned} \tag{43}$$

This is just the general fundamental solution presented by Westphal Jr. et al. [37], if we substitute here the constants $\alpha_1 \leftarrow \alpha\lambda$ and $\alpha_2 \leftarrow \beta\lambda$.

4. A connection between the Kirchhoff and the Reissner/Mindlin general fundamental solutions

Consider the general fundamental solutions for the Kirchhoff's plate operator [37],

$$U(r) = \frac{1}{8\pi D}[r^2 \ln(\alpha_K r) + F] \tag{44}$$

and Reissner/Mindlin, just presented above, where α_K and F are the free coefficients of Kirchhoff's plate model general fundamental solution.

We now look at the last member of Eq. (43), in the limit as the plate thickness h approaches zero and/or the transverse shear modulus coefficient k_G approaches infinity,

$$\lim_{\substack{h\to 0\\ k_G\to\infty}} U_{33}(r) = \lim_{\substack{h\to 0\\ k_G\to\infty}} \frac{1}{8\pi D}\left\{r^2\left[\ln(\alpha_1 r) - \frac{1}{2}\right] - \frac{2}{3k^2(1-\nu)}\frac{h^2}{k_G}\ln(\alpha_2 r)\right\}, \tag{45}$$

or

$$\lim_{\substack{h\to 0\\ k_G\to\infty}} U_{33}(r) = \frac{1}{8\pi D}\left[r^2\left(\ln(\alpha_1 r) - \frac{1}{2}\right)\right]. \tag{46}$$

Comparing Eqs. (44) and (46) we have, for

$$\ln(\alpha_K r) = \ln(\alpha_1 r) - \frac{1}{2}, \tag{47a}$$

$$F = 0, \tag{47b}$$

that

$$\lim_{\substack{h \to 0 \\ k_G \to \infty}} U_{33}(r) = U(r) . \tag{48}$$

From Eq. (47a)

$$\alpha_K = \alpha_1 \, e^{-1/2} , \tag{49}$$

and, substituting the Eqs. (47b) and (49) into Eq. (44), it results

$$U(r) = \frac{1}{8\pi D} r^2 \left[\ln(\alpha_1 r) - \frac{1}{2} \right] . \tag{50}$$

Considering now

$$\frac{\partial}{\partial x_\alpha} \left[\lim_{\substack{h \to 0 \\ k_G \to \infty}} U_{33}(r) \right] = \frac{1}{4\pi D} \ln(\alpha_1 r) r r_{,\alpha} , \tag{51}$$

and comparing with Eq. (43), we can verify that

$$\frac{\partial}{\partial x_\alpha} \left[\lim_{\substack{h \to 0 \\ k_G \to \infty}} U_{33}(r) \right] = U_{\alpha 3}(r) . \tag{52}$$

This comes to be exactly the fundamental solution for the rotations in the Kirchhoff plate model.

5. Particular fundamental solution

We can simplify the general fundamental solutions previously presented, Eqs. (43), (50) and (52). We just make the simplest possible choice for the remaining free coefficients to be

$$\alpha_1 := 1 , \qquad \alpha_2 := 1 , \tag{53}$$

finally resulting the particular fundamental solutions

- Kirchhoff plate model

$$\begin{aligned} U(r) &= \frac{r^2}{8\pi D} \left[\ln r - \frac{1}{2} \right] , \\ U_\alpha(r) &= \frac{\partial U(r)}{\partial x_\alpha} = \frac{1}{4\pi D} \ln(r) r r_{,\alpha} ; \end{aligned} \tag{54}$$

- Reissner and Mindlin plate models

$$\begin{aligned} U_{\alpha\beta}(r) &= \frac{1}{4\pi D(1-\nu)} \{ [4B(\lambda r) - (1-\nu) \ln r] \delta_{\alpha\beta} - [4A(\lambda r) + 1 - \nu] r_{,\alpha} r_{,\beta} \} , \\ U_{\alpha 3}(r) &= -U_{3\alpha}(r) = \frac{1}{4\pi D} \ln(r) r r_{,\alpha} , \\ U_{33}(r) &= \frac{1}{8\pi D} \left\{ r^2 \left[\ln r - \frac{1}{2} \right] - \frac{8}{(1-\nu)\lambda^2} \ln r \right\} . \end{aligned} \tag{55}$$

For a clamped circular plate of radius $r = a$ submitted to a unitary concentrated load applied at the center $r = 0$, the analytical solution for the transversal displacement u_3 is, according to Kirchhoff's plate model [30],

$$u_3 = \frac{1}{8\pi D} \left[r^2 \left(\ln r - \frac{1}{2} \right) + \left(\frac{1}{2} a^2 - r^2 \ln a \right) \right] . \tag{56}$$

Canceling the terms involving the radius a of the plate in the above solution, retaining consequently only the terms not related to its dimension in the brackets, we obtain exactly the fundamental solution $U(r)$ of the Kirchhoff plate model presented in Eq. (54).

The procedure just followed in analysing the above closed solution was considered by Hartmann [11], but he used instead the analytical solution for a *simply supported* plate and discarded the constant term $-(3+\nu)/(2(1+\nu))$ that should replace the term $-1/2$ when this problem is considered in obtaining the fundamental solution in question. However, the most right problem to be considered is just that of a *clamped* plate, since the boundary conditions of the auxiliary problem are those of vanishing displacements (transversal displacement and rotations), which are the variables of our starting Lamé/Navier system equation (1).

The fundamental solution for the Kirchhoff plate model with the factor $-1/2$ was also presented by Costa Jr. and Brebbia [6], where the factor in question was obtained through numerical experiments. This factor was here obtained through simple mathematical considerations.

The procedure here presented in determining general fundamental solutions seems to be very practical for another type of operators, specially for those involving basic variables of different nature, as is the present case. As the specialization of the final form of our sixth-order general solution is in close agreement with those of a thin clamped circular plate submitted to a centrally located unitary loading, we believe that our solution forms a part of the analytical solution of a Reissner and/or Mindlin plate under the same conditions.

6. Integral formulation and general tensors

The integral equations for Reissner's and Mindlin's plate models were already presented in several papers, the related bibliography at the end of this paper serving as an example for them. These equations are summarized next, and we present the respective general tensors with the free constants α_1 and α_2. P and Q are general domain points. When these points are located on the boundary they are denoted by p and q, respectively. The plate mid-surface domain is denoted by Ω and its Lipschitz continuous boundary by Γ.

The integral equations for the generalized displacements $u_i(P)$, Eq. (18b), and resultant stresses $\tau_{\alpha i}(P) \equiv \tau^{(1)}_{\alpha i}(P)$, Eq. (16), with $t_i(q) = \tau_{\alpha i}(q) n_\alpha(q)$, where $n_\alpha(q)$ denotes the components of the outward unit normal to Γ, are

$$\begin{aligned} c_{ij}(P)u_j(P) &+ \int_\Gamma T_{ij}(P,q)u_j(q)\,\mathrm{d}\Gamma(q) \\ &= \int_\Gamma U_{ij}(P,q)t_j(q)\,\mathrm{d}\Gamma(q) + \int_\Omega [U_{i3}(P,Q) - MU_{i\alpha,\alpha}(P,Q)]\sigma^{(1)}(Q)\,\mathrm{d}\Omega(Q) \end{aligned} \tag{57}$$

and

$$\begin{aligned} \tau_{\alpha i}(P) = &\int_\Gamma U_{\alpha ij}(P,q)t_j(q)\,\mathrm{d}\Gamma(q) - \int_\Gamma T_{\alpha ij}(P,q)u_j(q)\,\mathrm{d}\Gamma(q) \\ &+ \int_\Omega [U_{\alpha i3}(P,Q) - MV_{\alpha i}(P,Q)]\sigma^{(1)}(Q)\,\mathrm{d}\Omega(Q) + M\sigma^{(1)}(P)\delta_{\alpha i}\,. \end{aligned} \tag{58}$$

For uniformly distributed loads $\sigma^{(1)}(Q) \equiv cte$, the above domain integrals can be transformed to

$$\int_\Omega [U_{i3}(P,Q) - MU_{i\alpha,\alpha}(P,Q)]\sigma^{(1)}(Q)\,\mathrm{d}\Omega(Q) = \sigma^{(1)}\int_\Gamma [\Lambda_{i,\alpha}(P,q) - MU_{i\alpha}(P,q)]n_\alpha(q)\,\mathrm{d}\Gamma(q) \tag{59}$$

and

$$\int_\Omega [U_{\alpha i3}(P,Q) - MV_{\alpha i}(P,Q)]\sigma^{(1)}(Q)\,\mathrm{d}\Omega(Q) = \sigma^{(1)}\int_\Gamma [Y_{\alpha i\beta}(P,q) - MU_{\alpha i\beta}(P,q)]n_\beta(q)\,\mathrm{d}\Gamma(q)\,. \tag{60}$$

The variable M is defined as

$$M := \begin{cases} \dfrac{\nu_3 k_G}{(1-\nu)\lambda^2 k_E} & \text{Reissner plate model}\,, \\ 0 & \text{Mindlin plate model}\,. \end{cases} \tag{61}$$

The general expressions for the above tensors are given in the following, being completed with the displacement fundamental solution Eq. (43).

- Tensor $T_{ij}(r)$

$$\begin{aligned}
T_{\alpha\beta}(r) &= \frac{-1}{4\pi r}\{[4A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)][r_{,\beta}n_\alpha + r_{,n}\delta_{\alpha\beta}] \\
&\quad + [4A(\lambda r) + (1+\nu)]r_{,\alpha}n_\beta - 2[8A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)]r_{,\alpha}r_{,\beta}r_{,n}\}, \\
T_{\alpha 3}(r) &= \frac{\lambda^2}{2\pi}[B(\lambda r)n_\alpha - A(\lambda r)r_{,\alpha}r_{,n}], \\
T_{3\alpha}(r) &= \frac{-1}{4\pi}\{[(1+\nu)\ln(\alpha_1 r) + \nu]n_\alpha + (1-\nu)r_{,\alpha}r_{,n}\}, \\
T_{33}(r) &= \frac{-1}{2\pi r}r_{,n}.
\end{aligned} \tag{62}$$

- Derivatives of the tensor $U_{ij}(r)$

$$\begin{aligned}
U_{\alpha\beta,\gamma}(r) &= \frac{-1}{4\pi D(1-\nu)r}\{4\lambda r K_1(\lambda r)r_{,\gamma}\delta_{\alpha\beta} + [4A(\lambda r) + (1-\nu)][r_{,\alpha}\delta_{\beta\gamma} \\
&\quad + r_{,\beta}\delta_{\alpha\gamma} + r_{,\gamma}\delta_{\alpha\beta}] - 2[8A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)]r_{,\alpha}r_{,\beta}r_{,\gamma}\}, \\
U_{\alpha 3,\gamma}(r) &= -U_{3\alpha,\gamma}(r) = \frac{1}{4\pi D}[\ln(\alpha_1 r)\delta_{\alpha\gamma} + r_{,\alpha}r_{,\gamma}], \\
U_{33,\gamma}(r) &= \frac{1}{4\pi D}\left[r\ln(\alpha_1 r) - \frac{4}{(1-\nu)\lambda^2}\frac{1}{r}\right]r_{,\gamma}.
\end{aligned} \tag{63}$$

- Tensor $U_{\alpha ij}(r)$

$$\begin{aligned}
U_{\alpha\beta\gamma}(r) &= \frac{1}{4\pi r}\{[4A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)][r_{,\alpha}\delta_{\beta\gamma} + r_{,\beta}\delta_{\alpha\gamma}] + [4A(\lambda r) \\
&\quad + (1+\nu)]r_{,\gamma}\delta_{\alpha\beta} - 2[8A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)]r_{,\alpha}r_{,\beta}r_{,\gamma}\}, \\
U_{\alpha 3\beta}(r) &= \frac{\lambda^2}{2\pi}[B(\lambda r)\delta_{\alpha\beta} - A(\lambda r)r_{,\alpha}r_{,\beta}], \\
U_{\alpha\beta 3}(r) &= \frac{-1}{4\pi}\{[(1+\nu)\ln(\alpha_1 r) + \nu]\delta_{\alpha\beta} + (1-\nu)r_{,\alpha}r_{,\beta}\}, \\
U_{\alpha 33}(r) &= \frac{1}{2\pi r}r_{,\alpha}.
\end{aligned} \tag{64}$$

- Tensor $T_{\alpha ij}(r)$

$$\begin{aligned}
T_{\alpha\beta\gamma}(r) &= \frac{D(1-\nu)}{4\pi r^2}\{[4A(\lambda r) + 2\lambda r K_1(\lambda r) + (1-\nu)][n_\alpha\delta_{\beta\gamma} + n_\beta\delta_{\alpha\gamma}] \\
&\quad + [4A(\lambda r) + (1+3\nu)]n_\gamma\delta_{\alpha\beta} - [16A(\lambda r) + 6\lambda r K_1(\lambda r) + 2(1-\nu) \\
&\quad + \lambda^2 r^2 K_0(\lambda r)][(r_{,\alpha}n_\beta + r_{,\beta}n_\alpha)r_{,\gamma} + (r_{,\alpha}\delta_{\beta\gamma} + r_{,\beta}\delta_{\alpha\gamma})r_{,n}] \\
&\quad - 2[8A(\lambda r) + 2\lambda r K_1(\lambda r) + (1+\nu)][r_{,\alpha}r_{,\beta}n_\gamma + r_{,\gamma}r_{,n}\delta_{\alpha\beta}] \\
&\quad + 4[24A(\lambda r) + 8\lambda r K_1(\lambda r) + 2(1-\nu) + \lambda^2 r^2 K_0(\lambda r)]r_{,\alpha}r_{,\beta}r_{,\gamma}r_{,n}\}, \\
T_{\alpha 3\beta}(r) &= \frac{-D(1-\nu)\lambda^2}{4\pi r}[(2A(\lambda r) + \lambda r K_1(\lambda r))(r_{,\beta}n_\alpha + r_{,n}\delta_{\alpha\beta}) \\
&\quad + 2A(\lambda r)r_{,\alpha}n_\beta - 2(4A(\lambda r) + \lambda r K_1(\lambda r))r_{,\alpha}r_{,\beta}r_{,n}], \\
T_{\alpha\beta 3}(r) &= \frac{D(1-\nu)\lambda^2}{4\pi r}[(2A(\lambda r) + \lambda r K_1(\lambda r))(r_{,\alpha}n_\beta + r_{,\beta}n_\alpha)
\end{aligned} \tag{65}$$

$$+ 2A(\lambda r)r_{,n}\delta_{\alpha\beta} - 2(4A(\lambda r) + \lambda r K_1(\lambda r))r_{,\alpha}r_{,\beta}r_{,n}] ,$$

$$T_{\alpha 33}(r) = \frac{D(1-\nu)\lambda^2}{4\pi r^2}[(\lambda^2 r^2 B(\lambda r) + 1)n_\alpha - (\lambda^2 r^2 A(\lambda r) + 2)r_{,\alpha}r_{,n}] .$$

- Tensor $V_{\alpha i}(r)$

$$\begin{aligned} V_{\alpha\beta}(r) &= \frac{1-\nu}{2\pi r^2}(\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}) , \\ V_{\alpha 3}(r) &= 0 . \end{aligned} \tag{66}$$

- Tensor $\Lambda_{i,\alpha}(r)$

$$\begin{aligned} \Lambda_{\alpha,\beta}(r) &= \frac{r^2}{128\pi D}[(4\ln(\alpha_1 r) - 3)\delta_{\alpha\beta} + 2(4\ln(\alpha_1 r) - 1)r_{,\alpha}r_{,\beta}] , \\ \Lambda_{3,\alpha}(r) &= \frac{r}{128\pi D}\left[r^2(4\ln(\alpha_1 r) - 3) - \frac{32}{(1-\nu)\lambda^2}(2\ln(\alpha_2 r) - 1)\right]r_{,\alpha} . \end{aligned} \tag{67}$$

- Tensor $Y_{\alpha i\beta}(r)$

$$\begin{aligned} Y_{\alpha\beta\gamma}(r) &= \frac{-(1-\nu)r}{64\pi}\Big\{(4\ln(\alpha_1 r) - 1)(r_{,\alpha}\delta_{\beta\gamma} + r_{,\beta}\delta_{\alpha\gamma}) \\ &\quad + \left[\frac{4(1+3\nu)}{1-\nu}\ln(\alpha_1 r) - 1\right]r_{,\gamma}\delta_{\alpha\beta} + 4r_{,\alpha}r_{,\beta}r_{,\gamma}\Big\} , \\ Y_{\alpha 3\beta}(r) &= \frac{1}{8\pi}[(2\ln(\alpha_2 r) - 1)\delta_{\alpha\beta} + 2r_{,\alpha}r_{,\beta})] . \end{aligned} \tag{68}$$

$r_{,n} := r_{,\alpha}n_\alpha$.

In Eq. (57) for a point $P \in \Omega$

$$c_{ij}(P) := \delta_{ij} . \tag{69}$$

The system (57) is valid for a point $p \in \Gamma$ if we observe that the boundary integral in the LHS should be interpreted in the Cauchy principal value sense, and according to Fig. 2 the matrix $c_{ij}(p)$ is [32,36]

$$[C(p)]_{3\times 3} = \frac{1}{2\pi}\begin{bmatrix} Y + \frac{(1+\nu)}{4}Y_s & \frac{(1+\nu)}{4}Y_c & 0 \\ & Y - \frac{(1+\nu)}{4}Y_s & 0 \\ & & Y \end{bmatrix}_{\mathrm{Sym}} , \tag{70}$$

being

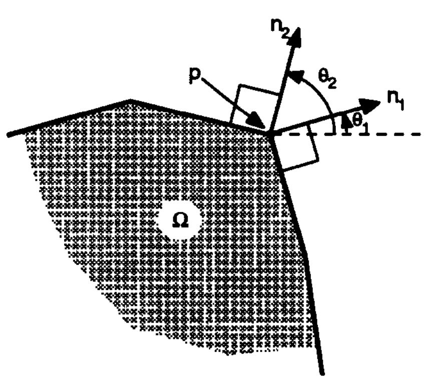

Fig. 2. Load point p on a corner point with normals (n_1, θ_1) and (n_2, θ_2).

$$
\begin{aligned}
Y &= \pi + \theta_1 - \theta_2 , \\
Y_s &= \sin(2\theta_2) - \sin(2\theta_1) , \\
Y_c &= \cos(2\theta_1) - \cos(2\theta_2) .
\end{aligned}
\tag{71}
$$

For a point p with continuous normals $\theta_2 = \theta_1$ we obtain

$$
c_{ij} = \frac{1}{2} \delta_{ij} . \tag{72}
$$

7. Numerical applications

In order to illustrate the use of the above equations, a BEM program was implemented. We tested a wide range of values for the free coefficients for some isotropic circular and rectangular plates and under some sets of boundary conditions and some choices of the number of integration points. It was confirmed that the results are weakly affected by using different coefficients. The results reported by de Barcellos and Westphal Jr. [9] are confirmed and some particularities of our code can be found there.

Here, we show only an example, where we consider a clamped circular plate submitted to a uniformly

Table 1
Boundary and internal results for a clamped circular plate uniformly loaded and discretized with 8 quadratic boundary elements

r/a	$\frac{64D}{qa^4} u_3$	$\frac{16D}{qa^3} u_n$	$\frac{16}{qa^2} \tau_{ss}$	$\frac{16}{qa^2} \tau_{nn}$	$\frac{2}{qa} \tau_{n3}$	IP
Exact	1.7314	0.0	1.3000	1.3000	0.0	
	1.7340	0.0	1.3015	1.3015	0.0	10
0.0	1.7309	0.0	1.2998	1.2998	0.0	20
	1.7305	0.0	1.2996	1.2996	0.0	40
Exact	0.0	0.0	−0.6	−2.0	−1.0	
	bc	bc		−1.9834	−0.9948	10
1.0	bc	bc	**	−1.9964	−0.9988	20
	bc	bc		−1.9981	−0.9993	40
Exact	0.0	0.0	−0.6	−2.0	−1.0	
	0.0	0.0	−0.5162	−1.9666	−0.9466	10
1.0*	0.0	0.0	−0.5120	−1.9666	−0.9500	20
	0.0	0.0	−0.4928	−1.9608	−0.9424	40

** Not calculated in this step.

Table 2
Percentual errors for 32 quadratic boundary elements

r/a	$\frac{64D}{qa^4} u_3$	$\frac{16D}{qa^3} u_n$	$\frac{16}{qa^2} \tau_{ss}$	$\frac{16}{qa^2} \tau_{nn}$	$\frac{2}{qa} \tau_{n3}$	IP
	5.3E-2	—	3.8E-2	3.8E-2	—	10
0.0	6.8E-3	—	5.0E-3	5.0E-3	—	20
	6.7E-4	—	5.1E-4	5.1E-4	—	40
	bc	bc		−0.4886	−0.4396	10
1.0	bc	bc	**	−6.5E-2	−5.9E-2	20
	bc	bc		−8.7E-3	−7.8E-3	40
	—	—	−1.0878	−0.1480	−0.5364	10
1.0*	—	—	−0.4344	−0.0668	−0.2190	20
	—	—	−0.3438	−0.0442	−0.1336	40

** Not calculated in this step.

distributed load and solve the hypersingular stress system for a boundary point. This possibility is apparently due to the symmetry of the problem, as the needed integration procedures are not implemented in our program.

7.1. Clamped circular plate under a uniformly distributed load

Here, we consider a model with 8 quadratic boundary elements and IP = 10, 20 and 40 integration points are used. The results for an isotropic plate are shown in Table 1. The local coordinates are n and s, the normal and tangential directions, respectively, and 'bc' denotes boundary conditions. We consider a plate according to Mindlin's model, of radius $a := 0.5$ and with $k^2 := 5/6$, $\nu := 0.3$, $h := 0.2$ and $q \equiv \sigma^{(1)}$.

The boundary conditions are $u_n = u_s = u_3 := 0$ (hard clamped [2]) and the calculated plate stresses on the boundary are $t_n = \tau_{nn}$, $t_s = \tau_{ns}$ and $t_3 = \tau_{n3}$. The plate stress τ_{ss} on the boundary cannot be directly calculated in solving the boundary system Eq. (57) without applying symmetry boundary conditions, discretizing only a quadrant of the plate. For internal points we define $\boldsymbol{n} := (1, 0)$ such that the local system coincides with the global one.

The values corresponding to the ratio r/a denoted by * were obtained for the hypersingular boundary stress system Eq. (58) at a boundary point, with the corresponding results multiplied by 2 (there appears a factor 1/2 multiplying the LHS of Eq. (58) for a boundary point, see [7]). Values of $\tau_{ss}(p)$ can be directly calculated in this way, without employing symmetry boundary conditions. These results are strongly influenced by our coarse discretization. To investigate this problem carefully, we solve the same problem, but employing instead 32 quadratic elements. The results, in terms of percentual errors, are shown in Table 2.

8. Conclusions

The general fundamental solution here presented was obtained through a concise and clear procedure. The considered material allowed to identify the contribution of two distinct types of transversal effects, namely transverse shear and direct deformations. It was proved that the fundamental solution of Kirchhoff's plate model can be reached if one of the two basic conditions satisfied by this model are met: thinness and/or very high transverse shear modulus.

In the final form of our fundamental solutions, Eqs. (43), (50) and (52), the two free coefficients always appear only as $\ln(\alpha_\beta r) \equiv \ln \alpha_\beta + \ln r$. The best choice is simply to select $\alpha_\beta := 1$.

High-order plate models are today a research area of ascending interest. Examples of some developments and corresponding numerical solutions by the FEM are the works of Schwab [26], Schwab and Wright [27], Li et al. [16,17]. Hencky–Bolle–Mindlin displacement-based plate models are considered in these works, being the nature of the formulation adequate to be solved by the FEM. Another possibility to develop two dimensional plate models is to start from a stress field as performed by Reissner (see also [19]). As the boundary integral equations involve a particular analytical solution of the problem, the fundamental solution, we do not encounter limitations on the nature of the basic field. Displacement- and stress-based models are handled with the same degree of difficulty.

The stress field of displacement-based models is determined by differentiation (through constitutive equations), whereas stress-based ones are obtained through integration of the equilibrium equations. Additionally, stress-based models lead to weighted averages of the displacement field that are generalizations of the corresponding field of displacement-based models. Classical or high-order Reissner plate models are consequently the best tools to analyse plate problems.

It is interesting to have a precise fundamental solution for the bi-harmonic problem because the two free coefficients C_4 and C_5 are exactly the same free coefficients present in high-order plate models (see [38]).

Acknowledgments

We would like to acknowledge L.H.M. Silva, who has previously developed some of the ideas presented here. TWJ and CSB are grateful for the financial support of the Deutsche Akademische Austauschdienst (DAAD) and the National Council for Scientific and Technological Development (CNPq/Brazil).

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