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# A fast *hp* adaptive finite element mesh design

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**Abstract**

An optimum *hp* adaptive mesh design is accomplished by minimizing the number of equations for a specified error limit. This new approach leads to a problem in which the *h* and *p* mesh parameters appear explicitly in the formulation. The optimal conditions yields a non-linear equation for each element which *simultaneously* supplies the optimum values for *h* and *p* parameters. The methodology here developed is applied to the numerical solution of several unidimensional elliptic boundary value problems. 

*Keywords:* *hp* Adaptivity; Finite element; Optimal mesh

## 1. Introduction

Despite the high degree of development reached by the finite element method in terms of mathematical theory and algorithms, the ability of defining an appropriate level of discretization for a given problem usually depends on the judgement of the analyst and on his previous experiences with similar problems. If the results are considered unsatisfactory, the discretization should be redone. Thus, it is reasonable to admit that if in the first try the expertise of the analyst failed, similar situation may happen when the results are being analyzed, that is, he/she may also fail in perceiving the quality of the results, or lack of it.

Due to such uncertainties, the possibility of automatically improving the numerical solution quality became an attraction center in computational mechanics. The approximated solution obtained by the finite element method can be improved by adaptive or feedback strategies which modify the solution representation where it is unsatisfactory. Such techniques are based on: node repositioning without affecting the mesh topology (*r*); mesh superposition (*s*); mesh refinement by change of element size (*h*); increase in the interpolation function orders (*p*); or combinations of them, mainly the last two (*hp*).

When the *h* and *p* strategies are combined, the most efficient method is obtained for a wide class of problems [2]. The main drawback of this class of adaptive methods is perhaps the complexity of the corresponding programming [4,8].

There exist several alternatives for starting an *hp* procedure in a finite element mesh. The first one perhaps the most used, consists in performing an *h* refinement in order to capture the eventual singularities present in the problem. Next, a *p* refinement is performed until a desired precision is reached. This

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technique, despite being simple, demands a high computational cost due to the considerable number of iterations usually required.

Another possible operational sequence is, based on some criterion, to make at each iteration a computation of new values for $h$ and $p$ parameters until an acceptable precision is reached. Among many works that follow this idea the papers of Guo and Babuška [6] and Rachowicz et al. [11] are worth mentioning.

The first one proposes a $p$ adaptive refinement for smooth solution regions and an $hp$ mesh sequence for regions containing singularities [1] *The meshes are geometrically graded toward the singular point with element degrees which are described by a nearly linearly increasing function starting in the second element away from the singularity. Further, the degree of the first element next to the singular point is greater than or equal to the degree of the second element.* This technique is known as: *true optimal hp mesh* [1]. An inconvenience of this methodology is that in the region of the domain where the solution is regular, a $p$ refinement may not be the best strategy option. Besides, since the mesh topology is preserved in this region, the possible solution to be obtained is strongly dependent on the initial discretization because it is not practical to enrich the finite element spaces by increasing the polynomial order indefinitely due to numerical instabilities.

The second mentioned work [11] is based on the error minimization for a fixed number of equations, where the $h$ and $p$ parameters are evaluated *independently*, following a convergence error analysis. This last strategy can be synthesized in the following conjecture [1]: *Between any two meshes in a sequence of optimal meshes, the change of the error per change in number of degree of freedom is maximized.* It is worth mentioning that in some circumstances this methodology leads to the so-called true optimal $hp$ meshes of Babuška et al. [1]. Nevertheless, the main inconvenience of this method is that a great number of iterations are needed in order to obtain a desired error level.

In this work, a new strategy based on a simultaneous computation of optimal $h$ and $p$ parameters is proposed. The formulation is based on the minimization of the number of degrees of freedom (d.o.f) for a constrained error level. The optimal conditions of this problem yield a non-linear equation for each element where $h$ and $p$ parameters are found in the an explicit form. The solution of this equation gives the optimal $h$ and $p$ mesh values (see the original work [7]).

Thus, in the proposed approach the user specifies a tolerance error and a short iterative process is started in order to find an optimal or quasi-optimal mesh for which the number of degrees of freedom is minimum for the pre-specified error.

In adaptive methods, the availability of local error estimation as a measure of the approximated solution quality is implicit [10]. Since this topic is presently out of the scope of this paper, only problems having analytical solutions are solved, the error being exactly computed, for illustrating the type of achievements that the proposed methodology can provide.

## 2. Elliptic boundary value problems

Consider an open bounded domain $\Omega$ in $\mathbb{R}^N$ with boundary $\Gamma = \Gamma_N \cup \Gamma_D$, where $\Gamma_N \cap \Gamma_D = \emptyset$. Assume that $\Gamma$ is smooth enough, i.e. a normal vector $\mathbf{n}$ exists almost everywhere (a.e.) on $\Gamma$. It is assumed there are displacement constraints on the boundary $\Gamma_D$, surface forces $\mathbf{t} \in L^2(\Gamma_N)$ and body forces $\mathbf{f} \in L^2(\Omega)$, where $L^2(\Omega)$ and $L^2(\Gamma_N)$ are the space of Lebesgue square-integrable functions over $\Omega$ and $\Gamma_N$, respectively.

When equilibrium statements are described in their variational form, it is possible to use weaker topological spaces than the ones used for strong formulations. Also, existence and uniqueness conditions are easier to be established, as well as a priori and a posteriori error estimators. Using formal mathematical expressions, elliptic boundary value problems may be written as follows:

Find $\mathbf{u} \in U$, such that

$$B(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{1}$$

where $B : U \times V \to \mathbb{R}$ is a bilinear symmetric operator and $l$ is a linear functional such that $l : V \to \mathbb{R}$, that is, $l \in V'$ where $V'$ is the dual space of $V$. The trial function space $U$, also called admissible functions space, is defined as

$$U = \{\mathbf{u} \in H^1(\Omega) \mid \mathbf{u} = \mathbf{g} \text{ in } \Gamma_D\}, \tag{2}$$

where $\mathbf{g}$ defines the function of prescribed values on $\Gamma_D$ and $H^1(\Omega)$ is a Hilbert space of order 1 on $\Omega$. The test functions space, or admissible variations space, is defined by

$$V = \left\{ \mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ in } \Gamma_D \right\}. \tag{3}$$

## 3. Finite element approximation

Most of the numerical methods used to solve boundary value problems rely on the definition of a finite-dimensional subspace $U_{hp}$ of the space of admissible functions $U$, in which the approximate solution $\mathbf{u}_{hp}$ is searched for. The finite element method is nothing more than a systematic and general procedure for constructing subspace families $U_{hp} \subset U$ and $V_{hp} \subset V$. In other words, it consists of solving the following approximate problem:

Find $\mathbf{u}_{hp} \in U_{hp} \subset U$, such that

$$B\left(\mathbf{u}_{hp}, \mathbf{v}_{hp}\right) = l\left(\mathbf{v}_{hp}\right) \quad \forall \mathbf{v}_{hp} \in V_{hp} \subset V. \tag{4}$$

In this case, the finite-dimensional space of admissible variations $V_{hp}$ and the finite-dimensional space of admissible functions $U_{hp}$ are equivalents, that is $U_{hp}$ and $V_{hp}$ are composed of identical collections of functions. By using this approach, the problem leads to a set of linear algebraic equations commonly written as

$$\mathbf{K}\overline{\mathbf{u}}_{hp} = \mathbf{F}, \tag{5}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{F}$ the generalized force vector and $\overline{\mathbf{u}}_{hp}$ is the discrete solution vector. The components of $\overline{\mathbf{u}}_{hp}$ represent the parameters of the linearly independent basis functions spanning $U_{hp}$.

According to the mathematical theory of the finite element method, the discretization error $\mathbf{e}$ depends on the domain partitioning and the choice of the finite element spaces [12], and may be defined as the difference between the exact value of $\mathbf{u}$ and the one numerically obtained $\mathbf{u}_{hp}$, i.e.

$$\mathbf{e} = \mathbf{u} - \mathbf{u}_{hp}. \tag{6}$$

Thus, the possibility of automatically improving the approximation $\mathbf{u}_{hp}$ through adaptive strategies is focus of special attention. Up to now, the most efficient way of error control for a wide class of problems is the *hp* adaptive technique.

## 4. An optimum *hp* mesh design

In this section, a technique to compute simultaneously optimal $h$ and $p$ parameters is shown. The idea is based on the minimization of the number of d.o.f for a given value measured in the $H^1$ norm. Mathematically, this can be written as

$$\text{minimize} \quad N_{\text{d.o.f.}}(h_n, p_n) = C \int_\Omega \rho(h_n, p_n)_\Omega \, \mathrm{d}\Omega, \tag{7}$$

$$\text{subject to} \quad \|\mathbf{e}(h_n, p_n)\|_{H^1(\Omega)} = \|\mathbf{e}_{\text{ad}}\|_{H^1(\Omega)}, \tag{8}$$

where $h_n$ and $p_n$ are the parameters of the new mesh, $\mathbf{e}_{\text{ad}}$ is the admissible approximation error, $N_{\text{d.o.f.}}(h_n, p_n)$ is the total number of degrees of freedom, $C$ is a positive constant value depending on the domain geometry, $\rho(h_n, p_n)$ is the density of degrees of freedom (d.o.f. density) and $\|\cdot\|_{H^1(\Omega)}$ denotes the norm in the $H^1(\Omega)$ space, which is equivalent to the energy norm in this context.

It is convenient to work in the optimization problem with equality constraints in order to allow both refinements as well as unrefinements of the mesh. This approach substantially simplifies the formulation.

Numerical experiments indicate that the d.o.f. density $\rho(h_n, p_n)$ is related to $h_n$ and $p_n$ parameters as follows:

$$\rho(h_n, p_n) = \left(\frac{p_n}{h_n}\right)^{\alpha}, \tag{9}$$

where $\alpha = 1, 2$ for 1D and 2D problems, respectively.

Rachowicz et al. [11] proved that, in some cases, an optimal point in $h$ refinement ($p = \text{const}$) is reached through uniform distribution of the error over the domain $\Omega$. Although this condition may not be related to optimality of $hp$ meshes in the sense of problem (7) and (8), it is often recognized as a very advantageous feature of the discretization [14]. Therefore, such equidistribution of the error will be considered as a basic premise, acting as an equality constraint. Let $m$ denote the total number of finite elements and $K$ the $K$th element of the mesh, the constraint (8) is now substituted by

$$\|\mathbf{e}(h_n, p_n)\|^2_{H^1(\Omega)} = \sum_K \left\|\mathbf{e}^K(h_n, p_n)\right\|^2_{H^1(K)} = m\left\|\mathbf{e}^K(h_n, p_n)\right\|^2_{H^1(K)} = m\left\|\mathbf{e}^K_{\text{ad}}\right\|^2_{H^1(K)}, \tag{10}$$

where, for each element $K$,

$$\left\|\mathbf{e}^K(h_n, p_n)\right\|_{H^1(K)} = \left\|\mathbf{e}^K_{\text{ad}}\right\|_{H^1(K)}. \tag{11}$$

Analogously, the objective function shown in Eq. (7) may also be stated in terms of the $K$th element as follows:

$$N_{\text{d.o.f.}}(h_n, p_n) = C\sum_K \int_K \rho(h_n, p_n)_K \, \mathrm{d}K. \tag{12}$$

The constant $C$ and the parameters $h_n$ and $p_n$ are strictly positive and so is $\rho(h_n, p_n)$. Thus, for sake of simplicity, the non-trivial global optimization problems (7) and (8) is now substituted by the set of local optimization problems

$$\text{minimize} \quad \rho(h_n, p_n)_K = \left(\frac{p_n}{h_n}\right)^{\alpha}, \tag{13}$$

$$\text{subject to} \quad \left\|\mathbf{e}^K(h_n, p_n)\right\|_{H^1(K)} = \left\|\mathbf{e}^K_{\text{ad}}\right\|_{H^1(K)}. \tag{14}$$

Hence, the proposed procedure consists in looking among the several possibilities of $hp$ enrichment for the one which aggregates the least amount of additional degrees of freedom. Of course, these two optimization statements (7) and (8) and (13) (14) are distinct and the relationship between their optimal solutions, if it exists, is an open question.

At this point, it is necessary to associate restriction (14) to the $h_n$ and $p_n$ parameters in order to obtain an expression relating them explicitly. An a priori estimate of interpolation error on element $K$ is given by Theorem 2.1 in the work of Oden et al. [10]. Moreover, it is well known that discretization errors in finite elements behave like interpolation errors except for pollutions and a constant value, also independent of control $h$ and $p$ parameters. Thus, an a priori discretization error can be expressed by the following theorem:

**Theorem 4.1.** *There exists a constant $C_1(r) > 0$, independent of element size $h$, order approximation $p$, error and solution $\mathbf{u} \in H^r(K)$, with $r > 1$, such that as $h \to 0$*

$$\left\|\mathbf{e}^K(h, p)\right\|_{H^1(K)} \leqslant C_1(r) h^{\mu-1} p^{-(r-1)} \|\mathbf{u}\|_{H^r(K)} + \text{pollution error}, \tag{15}$$

*where $\mu = \min(p + 1, r)$.*

Eq. (15) can also be written in terms of the error energy norm for the new $h_n$ and $p_n$ mesh parameters as

$$\left\|\mathbf{e}^L(h_n, p_n)\right\|_{H^1(L)} \leqslant C_2(r) h_n^{\mu_n - 1} p_n^{-(r-1)} \|\mathbf{u}\|_{H^r(L)} + \text{pollution error}, \tag{16}$$

where $\mu_n = \min(p_n + 1, r)$ and $L$ denotes the $L$th element of the new mesh.

If the mesh is fine enough, then the $\|\mathbf{u}\|_{H^r(L)}$ and $\|\mathbf{e}^L(h_n,p_n)\|_{H^1(L)}$ may be approximated, respectively, as

$$\|\mathbf{u}\|_{H^r(L)} \approx \gamma\|\mathbf{u}\|_{H^r(K)} \quad \text{and} \quad \left\|\mathbf{e}^L(h_n,p_n)\right\|_{H^1(L)} \approx \gamma\left\|\mathbf{e}^K(h_n,p_n)\right\|_{H^1(K)}, \tag{17}$$

where the factor $\gamma$ is given by $\gamma = (h_n/h)^{N/2}$.

By substituting Eqs. (17) in Eq. (16), the admissible error energy norm for the element $K$ takes the form

$$\gamma\left\|\mathbf{e}^K(h_n,p_n)\right\|_{H^1(K)} = \gamma\left\|\mathbf{e}^K_{\text{ad}}\right\|_{H^1(K)} \leqslant C_2(r)h_n^{\mu_n-1}p_n^{-(r-1)}\gamma\|\mathbf{u}\|_{H^r(K)} + \text{pollutions}, \tag{18}$$

Disregarding the pollutions and dividing Eq. (15) by Eq. (18), results

$$\frac{\|\mathbf{e}^K(h,p)\|_{H^1(K)}}{\|\mathbf{e}^K(h_n,p_n)\|_{H^1(K)}} = \frac{\|\mathbf{e}^K(h,p)\|_{H^1(K)}}{\left\|\mathbf{e}^K_{\text{ad}}\right\|_{H^1(K)}} = \beta(r)\frac{h^{\mu-1}}{h_n^{\mu_n-1}}\left(\frac{p}{p_n}\right)^{-(r-1)} = \xi, \tag{19}$$

where $\beta(r)$ is a constant of proportionality that depends on the regularity $r$. This parameter was introduced in order to allow the quotient between two convergence behavior (upper bounds of Eqs. (15) and (18)). Parameter $\xi$ relates the current error energy norm over the element $K$, $\|\mathbf{e}^K(h,p)\|_{H^1(K)}$, to the desired error energy norm over the same subdomain $K$, $\left\|\mathbf{e}^K_{\text{ad}}\right\|_{H^1(K)}$.

Note that the constants $C_1(r)$ and $C_2(r)$ are independent of the mesh parameters $h$, $p$, $h_n$ and $p_n$, but they depend on distortions of individual elements. Even so, they have been simplified in Eq. (19), i.e. it was assumed that $C_1(r) \approx C_2(r)$. From Eq. (19)

$$h_n = \left[\beta(r)\frac{h^{\mu-1}}{\xi}\left(\frac{p_n}{p}\right)^{(r-1)}\right]^{1/(\mu_n-1)}. \tag{20}$$

Finally, since the constraint uses parameters $h_n$ and $p_n$ explicitly, the constrained optimization problem of two variables may be rewritten as a one-dimensional unconstrained problem just by substituting Eq. (20) into the objective function (13). This operation leads to

$$\text{minimize} \quad \rho(p_n)_K = \left\{\frac{p_n}{[\beta(r)(h^{\mu-1}/\xi)(p_n/p)^s]^{1/(\mu_n-1)}}\right\}^{\alpha}, \quad \text{where } s = r-1. \tag{21}$$

The first order necessary optimality condition is that the first derivative of Eq. (21) with respect to the variable $p_n$ must be zero [3], i.e.

$$\frac{\mathrm{d}}{\mathrm{d}p_n}\rho(p_n)_K = 0. \tag{22}$$

However, it is still impossible to obtain the derivative in Eq. (22) because of the dependency of $\mu_n$ with regard to the regularity $r$ and the polynomial order $p_n$ for each different problem and the existence and intensity of singularities. Thus, some specific cases are now discussed, focusing different types of refinement, that is $h$, $p$ or $hp$, and also different regularity levels.

### 4.1. h Meshes

In this case, the polynomial order is considered constant, i.e. $p_n = p$, and an $h$ refinement is performed on the mesh. Therefore, Eqs. (19) and (20) take the forms:

(i) If $p > s$, then $\mu = s+1$, therefore

$$\xi = \beta(r)\left(\frac{h}{h_n}\right)^s \quad \text{or} \quad h_n = h\left(\frac{\beta(r)}{\xi}\right)^{1/s}. \tag{23}$$

(ii) If $p \leqslant s$, then $\mu = p+1$, thus

$$\xi = \beta(r)\left(\frac{h}{h_n}\right)^p \quad \text{or} \quad h_n = h\left(\frac{\beta(r)}{\xi}\right)^{1/p}. \tag{24}$$

This result was already obtained by Zienkiewicz and Zhu [13]. In fact, the optimization problem disappears because, satisfying simultaneously both conditions, $p_n = p$ and Eq. (19), variables $h_n$ and $p_n$ are uniquely determined. Therefore, to obtain an error equidistribution over the domain is enough to satisfy one of Eqs. (23) or (24) depending on the regularity of the problem.

### 4.2. p Meshes

In this case, the element size is fixed: $h_n = h$. Then, a $p$ adaptive refinement is done and Eqs. (19) and (20) take the following forms

(i) If $p > s$ and $p_n > s$, then $\mu = s + 1$ and $\mu_n = s + 1$, therefore

$$\xi = \beta(r)\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad p_n = p\left(\frac{\xi}{\beta(r)}\right)^{1/s}. \tag{25}$$

(ii) If $p \leqslant s$ and $p_n > s$, then, $\mu = p + 1$ and $\mu_n = s + 1$, thus

$$\xi = \frac{\beta(r)}{h^{(s-p)}}\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad p_n = p\left(\frac{\xi}{\beta(r)}\right)^{1/s} h^{(s-p)/s}. \tag{26}$$

(iii) If $p \leqslant s$ and $p_n \leqslant s$, then $\mu = p + 1$ and $\mu_n = p_n + 1$, therefore

$$\xi = \frac{\beta(r)}{h^{(p_n-p)}}\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad p_n = p\left(\frac{\xi}{\beta(r)}\right)^{1/s} h^{(p_n-p)/s}. \tag{27}$$

(iv) If $p > s$ and $p_n \leqslant s$, then $\mu = s + 1$ and $\mu_n = p_n + 1$, since

$$\xi = \frac{\beta(r)}{h^{(p_n-s)}}\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad p_n = p\left(\frac{\xi}{\beta(r)}\right)^{1/s} h^{(p_n-s)/s}. \tag{28}$$

It is easy to see that in this case, as well as in the last one, the almost constant distribution of error is guaranteed by satisfying one of Eqs. (25), (26), (27) or (28), depending upon the problem under analysis. Once again, the feasible region of the optimization problem has been reduced to a fixed point $h_n$ and $p_n$ satisfying constraint conditions.

### 4.3. hp Meshes

Here, the $hp$ refinement is finally discussed. In this case, variables $h_n$ and $p_n$ are not subject to another additional restriction expect for the original constraint of the optimization problem. Thus, from Eqs. (19) and (20) one has:

(i) If $p > s$ and $p_n > s$, then $\mu = s + 1$ and $\mu_n = s + 1$, therefore

$$\xi = \beta(r)\left(\frac{p_n h}{h_n p}\right)^s \quad \text{or} \quad h_n = \left(\frac{\beta(r)}{\xi}\right)^{1/s}\frac{p_n h}{p}. \tag{29}$$

(ii) If $p \leqslant s$ and $p_n > s$, then $\mu = p + 1$ and $\mu_n = s + 1$, since

$$\xi = \beta(r)h^p\left(\frac{p_n}{h_n p}\right)^s \quad \text{or} \quad h_n = \left(\beta(r)\frac{h^p}{\xi}\right)^{1/s}\frac{p_n}{p}. \tag{30}$$

In these first two cases, where the solution regularity is low ($s < p_n$), it is not possible to solve the optimization problem as proposed because when constraints (29) and (30) are substituted into the objective function (13) the result is an expression independent of the $h_n$ and $p_n$ variables. For these cases it is suggested, depending on the solution regularity, to fix variable $p_n$ and perform an $h$ refinement (Section 4.1) or vice versa, i.e. to fix the element size $h_n$ and perform a $p$ refinement (Section 4.2).

(iii) If $p \leqslant s$ and $p_n \leqslant s$, then $\mu = p + 1$ and $\mu_n = p_n + 1$, therefore

$$\xi = \beta(r)\frac{h^p}{h_n^{p_n}}\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad h_n = \left[\beta(r)\frac{h^p}{\xi}\left(\frac{p_n}{p}\right)^s\right]^{1/p_n}. \tag{31}$$

By substitution of these constraints in the objective function (13), one has

$$\rho(p_n)_K = \left\{ \frac{p_n}{[\beta(r)(h^p/\xi)(p_n/p)^s]^{1/p_n}} \right\}^{\alpha}. \tag{32}$$

(iv) If $p > s$ and $p_n \leqslant s$, then $\mu = s + 1$ and $\mu_n = p_n + 1$, thus

$$\xi = \beta(r)\frac{h^s}{h_n^{p_n}}\left(\frac{p_n}{p}\right)^s \quad \text{or} \quad h_n = \left[\beta(r)\frac{h^s}{\xi}\left(\frac{p_n}{p}\right)^s\right]^{1/p_n}. \tag{33}$$

Repeating the same procedure as before

$$\rho(p_n)_K = \left\{ \frac{p_n}{[\beta(r)(h^s/\xi)(p_n/p)^s]^{1/p_n}} \right\}^{\alpha}. \tag{34}$$

In the last two cases, it is possible to obtain the derivative of the objective functions (32) and (34) with regard to $p_n$ by using Eq. (22) in order to obtain a first order necessary condition for the optimal value of $p_n$. After identifying $p_n$, it may be substituted in Eqs. (31) and (33), depending on the case, and the new element optimal size $h_n$ is obtained.

One of the limitations of this procedure (and of others based on FEM convergence properties) is that the characteristics of the space to which the solution belongs should be known a priori. This enforces a strong dependence with the previous knowledge of the problem from the point of view of the user. In order to eliminate this recursive problem, it is possible estimate the regularity $r$ by analysis of the FEM convergence properties [9], that may be easily combined with the method here in development. However, an alternative technique is proposed as:

**Lemma 4.1.** *Let be* $p = s$, $\beta(r) = 1$, $\mu = p + 1$, $\mu_n = p_n + 1$ *and consider Eqs.* (19) *and* (20). *Then, one has*

$$\xi = \frac{1}{h_n^{p_n}}\left(\frac{p_n h}{p}\right)^p \quad \textit{or} \quad h_n = \frac{1}{\xi^{1/p_n}}\left(\frac{p_n h}{p}\right)^{p/p_n}. \tag{35}$$

**Proof.** Straightforward. □

Note that this modification on the problem constraint is quite severe. Strictly speaking, the assumptions of Lemma 4.1 enforce the condition $p_n \leqslant p$. This means that, under the adopted hypothesis, the new value of $h_n$ in Eq. (35) is consistent with only $p$ unenrichment process. This drawback is a consequence of fixing values for $\mu$ and $\mu_n$ due to the lack of knowledge of the regularity $r$. Despite this limitation, numerical tests will be performed using the procedure also in $p$ enrichment. Further, the hypotheses adopted here lead to the following lemma:

**Lemma 4.2.** *Consider the objective function* (13), *the result of Lemma* 4.1 *described by Eq.* (35) *and let* $p = s$, $\beta(r) = 1$, $\mu = p + 1$, $\mu_n = p_n + 1$. *Then, the d.o.f. density can be determined by*

$$\rho(p_n)_K = \left\{ \frac{p_n}{(1/\xi^{1/p_n})(p_n h/p)^{p/p_n}} \right\}^{\alpha} = \left\{ \frac{p_n \xi^{1/p_n}}{(p_n h/p)^{p/p_n}} \right\}^{\alpha}. \tag{36}$$

**Proof.** Straightforward. □

The objective function given by Eq. (36) establishes a relationship between the new polynomial order $p_n$ and the d.o.f density $\rho(p_n)_K$ which can be better understood through Figs. 1(a) and (b), for refinement and unrefinement, respectively.

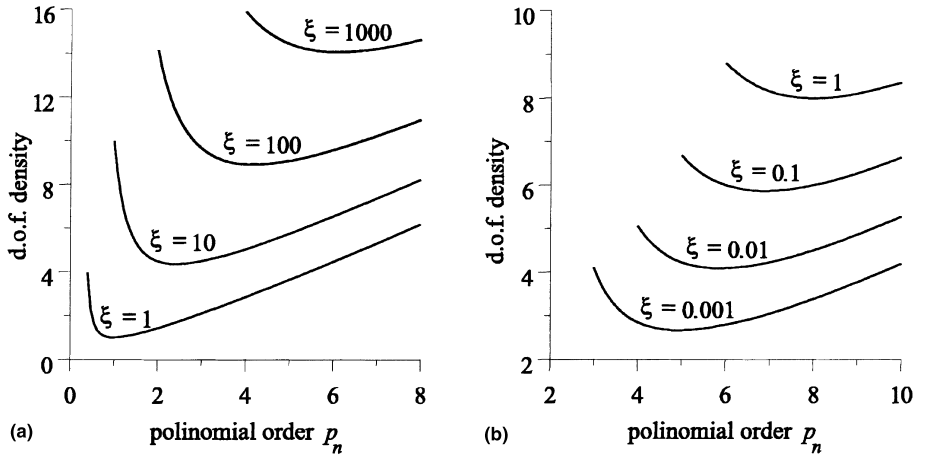


Fig. 1. Objective function for various values of $\xi$ and $h = 1$.

Finally, it can be shown that the optimization problem analyzed here consists in solving a non-linear equation, independent of $s$ and $\alpha$, in order to obtain optimal values for $p_n$ and $h_n$. This result is stated by the following theorem:

**Theorem 4.2.** *Let $p = s$, $\beta(r) = 1$, $\mu = p + 1$ and $\mu_n = p_n + 1$. Then, a necessary and sufficient optimality condition for the minimum $\rho(p_n)_K$ (Eq. (36)) is given by the following equation*:

$$p_n - p - \ln \xi + p \ln \left( \frac{p_n h}{p} \right) = 0 \quad or \quad p_n = \varphi(p_n), \tag{37}$$

*where $\varphi(p_n)$ is given by*

$$\varphi(p_n) = p + \ln \xi - p \ln \left( \frac{p_n h}{p} \right). \tag{38}$$

*Moreover, this optimality condition is independent of regularity $r$ (or $s$) of solution and the parameter $\alpha$ related to the physical dimension of the problem.*

**Proof.** By substituting Eq. (36) in Eq. (22), one has

$$\frac{\mathrm{d}}{\mathrm{d}p_n} \rho(p_n)_K = \left[ \frac{\alpha}{(p_n)^2} \right] \left[ \frac{p_n \xi^{1/p_n}}{(p_n h/p)^{p/p_n}} \right]^{\alpha} \left[ p_n - p - \ln \xi + p \ln \left( \frac{p_n h}{p} \right) \right] = 0. \tag{39}$$

Note that the first as well as the second factor of Eq. (39) provide a trivial solution for optimization problem, that is $p_n \to \infty$ or $\xi \to 0$. Thus, only the third factor, which is independent of $\alpha$ and $r$, allows the identification of a non-trivial solution, shown by Eq. (37).

Deriving Eq. (37) with respect to $p_n$ results in

$$1 + \frac{p}{p_n} > 0. \tag{40}$$

Therefore, due to convexity of Eq. (36), the theorem is proved. □

From this result, one can conclude that the polynomial order $p_n$ may be characterized as a fixed point of $\varphi(p_n)$. Thus, the optimization problem may be stated as solving the non-linear equation for each finite element

$$p_n - \varphi(p_n) = 0. \tag{41}$$

This result suggests the use of numerical methods in order to obtain a polynomial order $p_n$ which should be used later to compute a new size $h_n$ of the element through Eq. (35). A possible procedure to obtain $p_n$ is applying a relaxation method leading to the following algorithm:

*Fixed point algorithm*. Let be $p$ an initial polynomial order, tol a tolerance criterion and maxiter a maximum number of iterations. Then, for a given relaxation factor $w > 0$, an estimative for $p_n$ in iteration $j+1$ is

$$p_n^{j+1} = (1-w)p_n^j + w\varphi\left(p_n^j\right). \tag{42}$$

The stop criterion is satisfied when the tolerance tol is achieved or when $j = \text{maxiter}$.

As a matter of fact, this is an integer optimization problem. However, parameter $\overline{p}_n$ is defined as the integer number which is closest to the real value $p_n$. This number is effectively used for the $p$ enrichment of the interpolation functions on each element and also for the computation of $h_n$. This assumption is quite acceptable due to the smooth behavior shown by the objective function (see Fig. 1).

The optimal values of $p_n$, $\overline{p}_n$ and $h_n$ for some values of $\xi$ and $h = 1$ are shown in Table 1. It may be noted that when $\xi > 1$, the refinement is provided by appreciable variations of $p_n$ while $h_n$ is maintained around 0.43. In the case of unrefinement ($\xi < 1$), the values of $p_n$ diminish slowly. Otherwise $h_n$ shows greater variations when compared with the case of refinement ($\xi > 1$). Thereafter, the methodology here proposed has a marked tendency of producing accentuated $p$ refinements. This fact runs accordingly with mathematical theory of FEM for regular problems [12].

The last detail deserves special attention. The error energy norm is not the most convenient argument to work with error control because it is an absolute quantity. A more useful quantity is the relative error $\eta_{\text{ad}}$, defined as [13]:

$$\eta_{\text{ad}} = \frac{\|\mathbf{e}_{\text{ad}}\|_{H^1(\Omega)}}{\|\mathbf{u}\|_{H^1(\Omega)}}. \tag{43}$$

As the energy norm of the solution is usually not known, it must be written in terms of $\|\mathbf{u}_{hp}\|_{H^1(\Omega)}$. From Eqs. (6) and (1), the norm of the exact solution $\|\mathbf{u}\|_{H^1(\Omega)}$ may be written as

$$\|\mathbf{u}\|^2_{H^1(\Omega)} = \|\mathbf{u}_{hp} + \mathbf{e}\|^2_{H^1(\Omega)} = B\left(\mathbf{u}_{hp} + \mathbf{e}, \mathbf{u}_{hp} + \mathbf{e}\right) = B\left(\mathbf{u}_{hp}, \mathbf{u}_{hp}\right) + B(\mathbf{e}, \mathbf{e}) + 2B\left(\mathbf{u}_{hp}, \mathbf{e}\right). \tag{44}$$

Due to the orthogonality of the error $\mathbf{e}$ with respect to the discrete space of variations $V_{hp} \equiv U_{hp}$,

$$B\left(\mathbf{u}_{hp}, \mathbf{e}\right) = 0 \quad \forall \mathbf{u}_{hp} \in U_{hp} \equiv V_{hp}. \tag{45}$$

Hence, the norm of the exact solution is given by the following expression:

$$\|\mathbf{u}\|_{H^1(\Omega)} = \left(B\left(\mathbf{u}_{hp}, \mathbf{u}_{hp}\right) + B(\mathbf{e}, \mathbf{e})\right)^{1/2} = \left(\|\mathbf{u}_{hp}\|^2_{H^1(\Omega)} + \|\mathbf{e}\|^2_{H^1(\Omega)}\right)^{1/2}. \tag{46}$$

Substituting Eq. (46) into Eq. (43), the maximum global relative error is obtained as

$$\eta_{\text{ad}} = \frac{\|\mathbf{e}_{\text{ad}}\|_{H^1(\Omega)}}{\left(\|\mathbf{u}_{hp}\|^2_{H^1(\Omega)} + \|\mathbf{e}\|^2_{H^1(\Omega)}\right)^{1/2}}. \tag{47}$$

Table 1
Optimal values of $p_n$, $\overline{p}_n$ and $h_n$, for various $\xi$ and $h = 1$

| | Refinement ($\xi \geqslant 1$ and $p = 1$) | | | | | Unrefinement ($\xi \leqslant 1$ and $p = 8$) | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\xi$ | 1 | 10 | 100 | 1000 | 10 000 | 1 | 0.1 | 0.01 | 0.001 | 0.0001 |
| $p_n$ | 1.00 | 2.42 | 4.18 | 6.10 | 8.12 | 8.00 | 6.89 | 5.87 | 4.94 | 4.11 |
| $\overline{p}_n$ | 1 | 2 | 4 | 6 | 8 | 8 | 7 | 6 | 5 | 4 |
| $h_n$ | 1.00 | 0.447 | 0.447 | 0.426 | 0.410 | 1.00 | 1.19 | 1.47 | 1.88 | 2.50 |

Assuming that the error is uniform over all the elements of the mesh, Eq. (47) may be written as

$$\eta_{\mathrm{ad}} = \frac{\left(m\left\|\mathbf{e}_{\mathrm{ad}}^{K}\right\|_{H^1(K)}^2\right)^{1/2}}{\left(\left\|\mathbf{u}_{hp}\right\|_{H^1(\Omega)}^2 + \|\mathbf{e}\|_{H^1(\Omega)}^2\right)^{1/2}} = \frac{\sqrt{m}\left\|\mathbf{e}_{\mathrm{ad}}^{K}\right\|_{H^1(K)}}{\left(\left\|\mathbf{u}_{hp}\right\|_{H^1(\Omega)}^2 + \|\mathbf{e}\|_{H^1(\Omega)}^2\right)^{1/2}},$$
$$\left\|\mathbf{e}_{\mathrm{ad}}^{K}\right\|_{H^1(K)} = \eta_{\mathrm{ad}} \frac{\left(\left\|\mathbf{u}_{hp}\right\|_{H^1(\Omega)}^2 + \|\mathbf{e}\|_{H^1(\Omega)}^2\right)^{1/2}}{\sqrt{m}}. \tag{48}$$

Finally, the admissible error energy norm of the $K$th element $\left\|\mathbf{e}_{\mathrm{ad}}^{K}\right\|_{H^1(K)}$ may be computed from the value of the $\eta_{\mathrm{ad}}$ parameter, usually provided by the user.

## 5. An *hp* adaptive refinement

One of the tasks of the *hp* adaptive refinement is the definition of convenient meshes, which relates adaptivity with automatic mesh generation methods [5]. Two main branches are found in the adaptive mesh generation techniques. The first one is based on element partitions [11] and the other one is focused on partial and even total mesh regeneration [13]. The former techniques have, as advantage, a low cost for mesh construction and solution projection from previous iterations. On the other hand, the geometry approximation is fixed by the first discretization and, in the case of having low levels of error, unrefinement is usually avoided.

Remeshing techniques have opposite properties. Geometry approximation improves as well as the mesh and unrefinement is easily performed. Meshing cost increases, but taking into account the performance of actual meshing algorithms this drawback may sometimes be accepted. From these considerations, total remeshing technique is here proposed as an appropriate choice within this context, although the present formulation is not restricted to this scheme.

It is important to mention that the primal variable $\mathbf{u}$ is here approached by using hierarchical base built from the integral of Legendre polynomials [12]. However, this technique usually produces non-conforming meshes in 2D and 3D problems that must be conveniently treated a posteriori, restraining the use of conventional FEM codes [8].

The choice of the strategy to be used for an *hp* refinement basically depends on the regularity of the solution $\mathbf{u}$. For singular solutions, the value of $s$ is assumed to exist within the range $0 < s < 1$. For smooth solutions, the $s$ parameter takes the value $s = p \geqslant 1$ (see Theorem 4.2).

### 5.1. Regular problems ($s \geqslant 1$)

In this case, an algorithm for total remeshing is used (see [5]). A new polynomial order $p_n$ (Eq. (37)) and a new element size $h_n$ (Eq. (35)) is associated to each element of the old mesh. From this information a new number $N_l \geqslant 1$ of elements within the domain is computed. This is performed by obtaining the closest integer to the real value $A_l$, that is

$$A_l = \int_0^1 \frac{1}{h_n(x)}\,\mathrm{d}x. \tag{49}$$

The position of the new nodes $i$, with $1 \leqslant i \leqslant N_l$, may be determined by finding $\lambda_i$ such that

$$i = \int_0^{\lambda_i} \frac{1}{h_n(x)}\,\mathrm{d}x, \tag{50}$$

where the nodes $i = 0$ and $i = N_l$ are initial and final nodes, respectively.

Once the new mesh is defined, a projection of the polynomial order $p_n$ from each element of the old mesh to the new one is done.

Adaptive remeshing has the drawback that it is not possible to reuse the linear equation system from one iteration to the other. However, this situation is counterbalanced by the fact that only few iterations are needed to reach an error level when using the *hp* calculus technique here proposed. This statement is further discussed in Section 6, with the aid of some numerical results. But one must remark that the scheme here proposed is also applicable to nested meshes in order to take advantage of previous computation.

### 5.2. Singular problems ($0 < s < 1$)

Singular problems usually solved with conventional FEM codes is such that $0 < s < 1$. The proposed formulation cannot be directly applied to problems involving cracks, vertexes, discontinuous loads, geometrical property variations, etc. Therefore, some special treatment is needed for these cases. This happens due to the hypothesis of Theorem 4.2, i.e. $p = s$, being too strong.

A possible approach is to use the so-called true *hp* optimal meshes [1] at the singular points certainly, the best error control technique for this class of problem. The meshes are based on a geometrical growth of the elements from the singular point. Thus, the mesh used here follows this refinement criterion:

$$h_n\left(1 + Q + Q^2 + \cdots + Q^{N_c}\right) = h. \tag{51}$$

$h$ being the actual size of the element next to the singular point, $h_n$ the new element size on the singularity and $N_c$ the number of geometrical layers. In this context, $h_n$ is evaluated by considering a pure $h$ refinement (Eq. (23)) and, for a given geometric progression ratio $Q > 1$,

$$N_c = \frac{\log\left[1 + (h/h_n)(Q-1)\right]}{\log Q} - 1. \tag{52}$$

As it is intended to extend these results for two- and three-dimensional problems, a fixed ratio $Q = 4$ is adopted for preventing high mesh distortion, although the ratio $Q = 5.88$ [2] would be preferable if a robust mesh generation program were available. Further, the regularity of the solution $s$ (or $r$) is fixed at $s = 1/2 < p$. This last important assumption comes from a pragmatic reasoning: it is assumed that an engineer analyst will know where the singularity is but not its intensity.

The polynomial order distribution around the singular point is defined by a linear growth law starting from the second element adjacent to the singularity, such that $\{p_n\} = \{1, 2, \ldots, N_c\}$. For the closest element to the singular point, $p_n = N_c + 1$ is chosen. In other words, the polynomial distribution on the singular region is $\{p_n\} = \{N_c + 1, 1, 2, \ldots, N_c\}$.

### 5.3. An hp refinement algorithm

The *hp* refinement strategy proposed here allows refinement as well as unrefinement. In the latter case, the polynomial order may vary between the range $1 \leqslant p_n \leqslant 8$ within one iteration. On the other hand, the element size has a growth limit to avoid mesh distortions. When $h_n > 1.3h$, the following projection is used [5]:

$$h_n := \frac{2hh_n}{h + h_n}. \tag{53}$$

A possible way to achieve the admissible error $\eta_{\mathrm{ad}}$ (Eq. (43)) in the optimization problem consists of dividing $\eta_{\mathrm{ad}}$ in a set of incremental steps. Thus, a new parameter, $\overline{\eta}_{\mathrm{ad}}$, is defined as

$$\overline{\eta}_{\mathrm{ad}} = (\eta_{\mathrm{ad}})^{\sqrt{\mathrm{iter/niter}}}, \tag{54}$$

niter being the total number of iterations within the adaptive procedure and iter the iter-th iteration. Therefore, $\overline{\eta}_{\mathrm{ad}}$ is considered as the limit of the admissible error on each iteration iter. In short, the proposed procedure may be summarized in the following steps:

1. Definition of $\eta_{\mathrm{ad}}$ and niter.
2. While iter $\leqslant$ niter, start adaptive procedure, i.e. while $\overline{\eta}_{\mathrm{ad}} \geqslant \eta_{\mathrm{ad}}$:

(a) The elements in connection with singular points are identified and a true optimal *hp* mesh refinement is performed on them, as presented in Section 5.2.
(b) On the other regions, a regular solution is assumed to exist. Thus, a direct and simultaneous computation of $h$ and $p$ parameters is done for each element following the technique developed in Section 4.3.
(c) Finally, a total remeshing procedure in accordance with new values of $h$ and $p$ is started as proposed in Section 5.1.

3. When iter = niter, one has $\overline{\eta}_{\mathrm{ad}} = \eta_{\mathrm{ad}}$ and, after a last analysis, the process stops. At this point, it is expected that the specified error level has been reached.

## 6. Numerical results

Some numerical tests were used to analyze the performance of the proposed technique. To this aim, the following elliptic boundary value problem is solved:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + f(x) = 0, \quad x \in \Omega = (0,1), \quad u(0) = g_0 \quad \text{and} \quad u(1) = g_1, \tag{55}$$

where $f(x)$ is specially chosen in order to produce great local variations on the solution $u(x)$ due to singularities or high gradients. Smooth solutions, as it is well known, lead to pure $p$ refinement.

All cases start from a uniform mesh with 10 linear elements, and the *hp* adaptive process is carried out considering that the exact solution $u(x)$ is known in advance in order to compute the error exactly.

In each of the following examples the proposed *hp* strategy is compared with the one proposed by Rachowicz et al. [11] as far as convergence rates and number of iterations to achieve a given level of error are concerned.

**Example 1.** In this example it is considered $g_0 = 0$, $g_1 = 0$ and $f(x)$ is obtained by

$$f(x) = 2\alpha \left[ \frac{1}{1 + \alpha^2(x - x_0)^2} + \frac{\alpha^2(x - x_0)(1 - x)}{\left[1 + \alpha^2(x - x_0)^2\right]^2} \right]. \tag{56}$$

Then, the exact solution $u(x)$ is given by

$$u(x) = (1 - x)\left[\tan^{-1}\alpha(x - x_0) + \tan^{-1}\alpha x_0\right]. \tag{57}$$

For $x_0 = 4/9$ and $\alpha = 50$ the function $u(x)$ is smooth, but presents high gradients near $x = x_0$.The convergence rates of the present formulation for 3, 4 and 5 iterations and the ones described in [11] for about 18 iterations are shown in Fig. 2. Comparing the curves, a similar convergence rate is observed. However, in this work only a few iterations are needed to reach a similar error level.

In Fig. 3, the *hp* mesh obtained after an iterative procedure with niter = 4 is shown. In this case, the refinement, as expected, is concentrated around the point $x = 4/9$ where high solution gradients are found.

**Example 2.** The same problem presented in Example 1 is analyzed here, but with $\alpha$ being equal to 200. The solution $u(x)$ is still smooth but presents higher gradients on $x = 4/9$.

Convergence rates of Rachowicz et al. [11] for about 18 iterations and of the present formulation for 3, 4 and 5 iterations are shown in Fig. 4. It can be noted that, on the beginning of the process, convergence rates are similar. However, due to the remeshing strategy, there exists a strong tendency of locating nodes near to the point $x = 4/9$ as it happened. Therefore, higher convergence rates appears. Finally, it must be mentioned once again that those values of error were obtained with only a few iterations.The final *hp* mesh for niter = 4 is shown in Fig. 5. In this case, a concentration of nodes near $x = 4/9$ is evident. Note that this technique has positioned a node close to the point $x = 4/9$ and this carried out such high convergence rates shown in the Fig. 4.

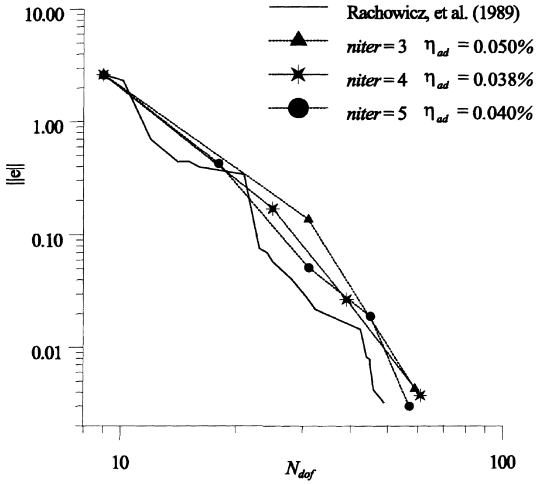


Fig. 2. *hp* Convergence (Example 1).

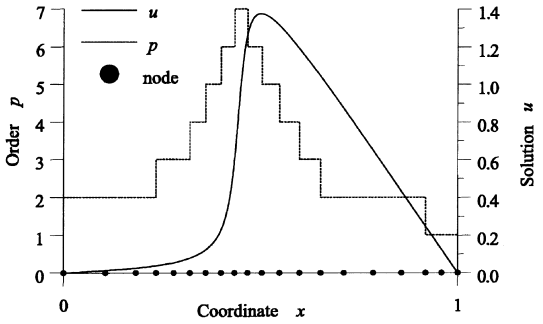


Fig. 3. Final *hp* mesh for niter = 4 (Example 1).

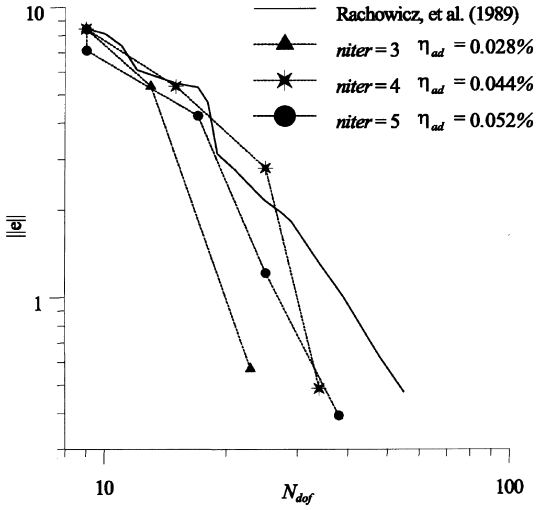


Fig. 4. *hp* Convergence (Example 2).

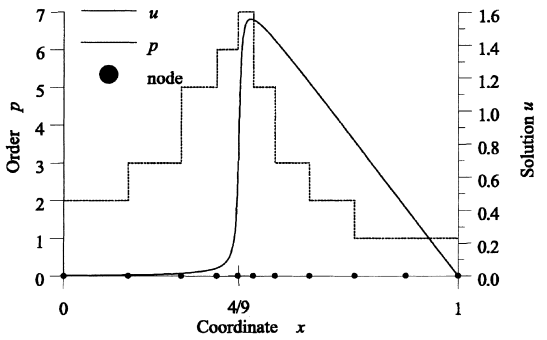


Fig. 5. Final *hp* mesh for niter = 4 (Example 2).

**Example 3.** In this last example, $g_0 = 0$, $g_1 = 1$ and the source term $f(x)$ is

$$f(x) = \alpha(\alpha - 1)x^{\alpha-2}. \tag{58}$$

Hence, the exact solution is

$$u(x) = x^{\alpha}. \tag{59}$$

Here $\alpha$ is selected to be 0.6 and a singular point is located at $x = 0$. Nevertheless, the energy norm of the solution is bounded, i.e.

$$\|u\|_{H^1(\Omega)} \equiv \left\| \frac{\mathrm{d}u}{\mathrm{d}x} \right\|_{L^2(\Omega)} = \sqrt{\int_0^1 \left( \frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x} = \frac{\alpha}{\sqrt{2\alpha - 1}} < \infty \quad \text{if } \alpha > \frac{1}{2}. \tag{60}$$

With the singular solution at hand, the Babuška et al. technique [1] is used next to the singular point as detailed in Section 5.2. On the remaining domain, the methodology proposed here is applied. A comparison between the results obtained by Rachowicz et al. [11] for about 60 iterations and by the present formulation for 3, 4 and 5 iterations are presented in Fig. 6. Once again, similar convergence rates can be noted, but there is an important difference in the required number of iterations to achieve a similar accuracy level.The final mesh for niter = 4 is depicted in Fig. 7. A detail of the refinement close to the singularity is shown in Figs. 8 and 9.

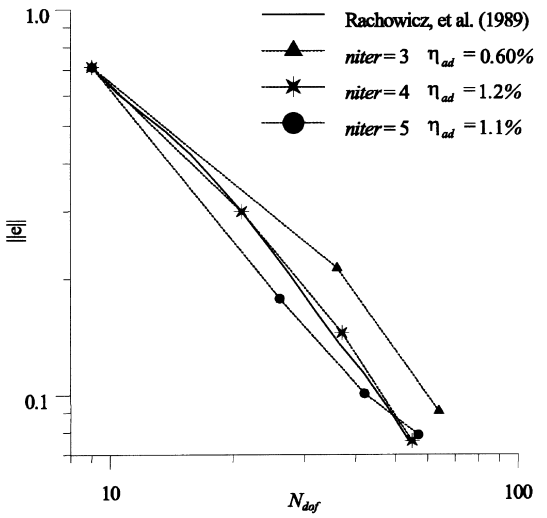


Fig. 6. *hp* Convergence (Example 3).

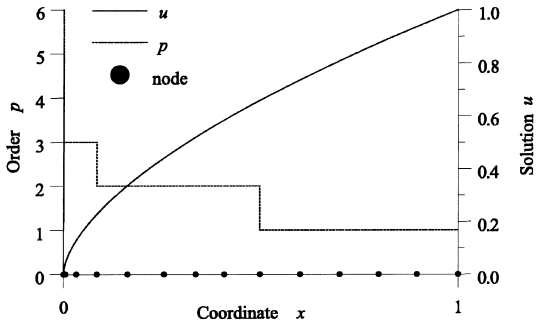


Fig. 7. Final *hp* mesh for niter = 4 (Example 3).

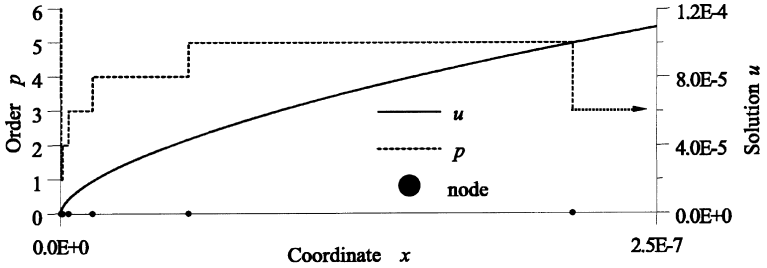


Fig. 8. Final *hp* mesh for niter = 4: zoom of $4 \times 10^6$ times (Example 3).

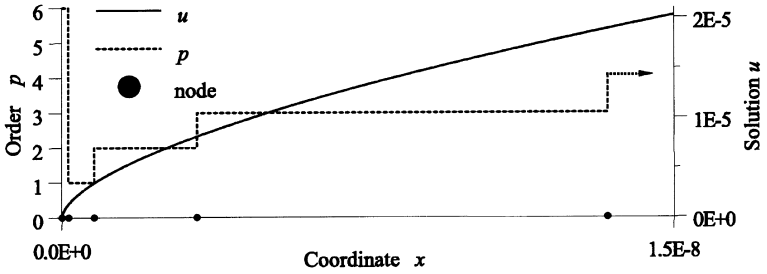


Fig. 9. Final *hp* mesh for niter = 4: zoom of $6.67 \times 10^7$ times (Example 3).

## 7. Conclusions

A new *hp* adaptive strategy based on the local minimization of the number of equations for a given error level is proposed here. This formulation leads to a non-linear one-variable equation for each element that may easily be solved by a fixed point algorithm. As a result, the new *h* and *p* parameters are simultaneously obtained at each element of the mesh for the next iteration.

Since the proposed formulation is independent of an adopted mesh refinement technique, a *remeshing* procedure was chosen here. *Remeshing* allows a complete independence of the discretization at each iteration. This strategy diminishes the responsibility of the analyst with regard to the previous knowledge of the problem behavior. On the other hand, computational costs are greater than the ones obtained with, for example, nested meshing because, in this last option, it is possible to reuse information of previous iterations. The disadvantage of nested schemes is that the new mesh strongly depends on the initial one and that unrefinements are not easy to perform.

The goal of this work is to propose a formulation applicable to 2D and 3D boundary value problems with the minimum previous knowledge about the topology of the spaces in which the solution is inserted. In

order to satisfy these requirements, many hypotheses are assumed. Therefore, the result of Theorem 4.2 may not coincide with the optimal meshes in the sense of Babuška et al. [1]. In fact, numerical experiments have shown that the algorithm usually leads to a mesh where the error is overestimated. Despite this problem, the obtained results are quite satisfactory in relation to the independence of the analyst in the adaptive process, to the convergence rates and, above all, to the reduction of the number of iterations to achieve an specified error level. In addition, the numerical results obtained with only four iterations are as good as those obtained through the conventional procedure by using four or five times more iterations for the regular problems (see Examples 1 and 2) and 10 or 12 times more iterations for the singular case (see Example 3).

Finally, it is worth mentioning that the necessary and sufficient optimality conditions (Theorem 4.2) are stated in a local sense, that is, at each element. The consideration of these conditions in a global sense, that is for all the mesh, is a question that still remains open. Moreover, other cost functions could have been considered instead and are currently under investigation.

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