ELSEVIER

PII: S0955-7997(96)00028-8

Engineering Analysis with Boundary Elements 17 (1996) 279–285

0955-7997/96/$15.00

On general fundamental solutions of some linear elliptic differential operators

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The derivation of general fundamental solutions of differential operators on tensor fields is converted, through Hörmander's method, in search of general fundamental solutions of operators on scalar fields. One resorts to the theory of distributions in order to guarantee the existence of the generalized functions required in the formulation. The procedure is applied in the determination of general fundamental solutions of some well known linear elliptic differential operators of the continuum mechanics. The study concludes that the use of general fundamental solutions can be computationally advantageous. Copyright © 1996 Elsevier Science Ltd

Key words: Boundary element method, fundamental solutions, Hörmander's method, plate bending, elasticity.

1 INTRODUCTION

The large applicability of the boundary element method (BEM) to solve engineering problems depends directly on the availability of fundamental solutions.[1,2] Although fundamental solutions are extensively described in a great number of publications, their derivation and general expressions are scarcely ever discussed. The goal of this paper is to discuss a well known procedure for the determination of general fundamental solutions of some basic linear elliptic differential operators of continuum mechanics.

At the present stage of development, the advanced application of BEM to particular problems has shown great dependence of correct interpretation and clever use of fundamental solutions in its general aspects. In this paper it is shown that fundamental solutions are formed by a combination of essential and complementary elementary functions.[3,4] Essential fundamental solutions were used extensively up to now. Nevertheless, little attention has been given to the complementary terms of fundamental solutions. This work is an attempt to cover this gap, and does not intend to be conclusive, in the sense of giving the best coefficients (or range of coefficients) of complementary functions. It should be noted that all fundamental solutions have complementary functions.[3]

The operators treated here are: Laplace, bi-harmonic (Kirchhoff's plate model), Reissner/Mindlin plate model, two- and three-dimensional elasticity. Although only applied to some operators, the procedure outlined here can be easily extended to all the family of linear elliptic differential operators with constant coefficients.

The indicial notation will be extensively used throughout this paper, with subscript greek indices in the range 1,2 and subscript roman indices in the range 1,2,3.

2 ABSTRACT THEORETICAL FOUNDATION

The establishment of the integral equations for a given physical phenomenon, starting from their mathematical description in differential form, is best performed through the application of the weighted residual method.[1] This method states that the weighting functions, according to which residual errors are minimized, are given by[1]

$$
\begin{aligned}
u_i^*(Q) &= U_{ji}(P,Q)e_j(P)\\
t_i^*(Q) &= T_{ji}(P,Q)e_j(P)
\end{aligned}
\qquad (1)
$$

where $u_i^*(Q)$ and $t_i^*(Q)$ are the generalized displacements and tractions, respectively, at the field point Q

corresponding to the fundamental solution; $U_{ji}(P,Q)$ and $T_{ji}(P,Q)$ represent, respectively, the displacements and tractions in the i direction at the field point Q corresponding to a generalized unit-concentrated load acting in the j direction applied at the load point $P(e_j(P))$.

Consider a problem described by its differential form

$$L_{ij}(\partial_Q)u_j(Q) = -F_{ij}(\partial_Q)q_j(Q) \tag{2}$$

where $L_{ij}(\partial_Q)$ is a linear elliptic differential operator with constant coefficients; $F_{ij}(\partial_Q)$ is any differential operator; ∂_Q means that the differential operators are applied at the field point Q; $q_j(Q)$ is a known vector; and, $u_j(Q)$ refers to the generalized basic variables (e.g. the generalized displacements in structural mechanics) of the problem in question.

Writing the above equations for the fundamental solution results in[1]

$$L_{ij}(\partial_Q)U_{kj}(P,Q) = -\delta(P,Q)\delta_{ik} \tag{3}$$

where $\delta(P,Q)$ is the Dirac's delta generalized function (Dirac's delta distribution[5]) applied in the load point P, and δ_{ik} is the Kronecker's delta symbol.

The Hörmander's method is applied in order to transform the original problem on a tensor field into another equivalent one on a scalar field. The scalar solution $G(P,Q)$ is related to the tensor solution $U_{kj}(P,Q)$ by[6]

$$U_{kj}(P,Q) = L_{kj}^{co}(\partial_Q)G(P,Q) \tag{4}$$

where $L_{kj}^{co}(\partial_Q)$ is the cofactor matrix of $L_{kj}(\partial_Q)$.

Substituting eqn (4) into (3), observing the equality

$$L_{ij}^{co}(\partial_Q)L_{kj}(\partial_Q) = L_{ij}(\partial_Q)L_{kj}^{co}(\partial_Q) = |\mathbf{L}|\delta_{ik} \tag{5}$$

where $|\mathbf{L}| \equiv \det L_{ij}(\partial_Q)$ is the determinant of $L_{ij}(\partial_Q)$, then it applies for

$$|\mathbf{L}|G(P,Q) = -\delta(P,Q) \tag{6}$$

Now, the problem of finding a tensor fundamental solution of the operator $L_{ij}(\partial_Q)$ was reduced to the search of the scalar fundamental solution of the operator $|\mathbf{L}|$. This is known as the Hörmander's method.[7]

Fundamental solutions have radial symmetry, since one is dealing with isotropic materials. Then, $G(P,Q) = G(Q,P) = G(r)$, where $r = \| P - Q \|$, that is, r is the distance between points Q and P, measurable according to the L^2-norm.

In usual engineering problems, such as those treated in this work, the operators considered are linear elliptic differential equations with constant coefficients. The homogeneous solutions are then easily obtained.

In order to determine $G(r)$, first the homogeneous equation corresponding to eqn (6) must be solved. After this, the theory of distributions is applied in order to determine the general fundamental solution, establishing that[4]

$$\langle G(r), |L|\phi(r)\rangle = -\phi(0) \qquad \forall\phi \in \mathscr{D}(\mathscr{R}^n) \tag{7}$$

where $\langle\cdot,\cdot\rangle$ denote the duality pairing between the spaces $\mathscr{D}(\mathscr{R}^n)$ and $\mathscr{D}'(\mathscr{R}^n)$; $\phi(r)$ is a test function; $\mathscr{D}(\mathscr{R}^n)$ is the space of test functions in $\mathscr{R}^n$; and $\mathscr{D}'(\mathscr{R}^n)$ is the space of distributions (the dual space of $\mathscr{D}(\mathscr{R}^n)$). Integrating by parts, after excluding a sphere S(P,ε) centered at the load point P with radius ε, a system of equations is obtained, one, which satisfying the above expression, gives the general fundamental solution $G(r)$. $G(r)$ is formed by a combination of essential and complementary elementary functions. The essential ones are those necessary to satisfy eqn (7), that is, they completely characterize the singular behavior, while the complementary ones do not affect this equation.

3 APPLICATION TO SOME OPERATORS

Following the procedure outlined above, some general fundamental solutions of linear elliptic differential operators concerning the range of interest of continuum mechanics will be considered.

3.1 Laplace operator

The Poisson equation

$$\Delta u = -b \tag{8}$$

is governed by the Laplace operator $\Delta(\cdot) = \partial^2(\cdot)/\partial x_i \partial x_i$. The Laplace equation corresponds to $b = 0$.

The fundamental solution, in agreement with the above presentation, is governed only by the operator $|\mathbf{L}|$. The right hand side of eqn (2), which in this case is related to Poisson's equation, is answerable only for the domain integral, when the problem is expressed in integral form. Then, to determine the fundamental solution one needs only to consider the Laplace equation

$$\Delta u = 0 \tag{9}$$

It can be seen that the differential operator governing the problem in question is given (formally) by

$$\mathbf{L} = \Delta \tag{10}$$

The correlation between the terms of the particular eqn (8) and those of the general eqn (2) can be directly expressed. The method outlined in the preceding section can be applied in order to determine the fundamental solution. Due to the fact that this problem is already expressed on a scalar field, Hörmander's method need not be applied, and in order to use the previous symbolism, the function $U(r)$ must be equalized to $G(r)$.

In this case, eqn (3) is expressed as

$$\Delta U(r) = -\delta(r) \tag{11}$$

whose homogeneous solution is, for the two-dimensional case

$$U(r) = C_1 \ln(r) + C_2 \tag{12}$$

and, for the three-dimensional case,

$$U(r) = C_1 \frac{1}{r} + C_2 \tag{13}$$

Now, it is only necessary to guarantee that the above solutions satisfy eqn (7).

Substituting Laplace's operator (eqn (10)) in eqn (7), integrating by parts through the use of Green's formula, and observing again that $U(r)$ is equal to the fundamental solution $G(r)$, one obtains

$$\lim_{\varepsilon \to 0}\left\{ - \int_{\Gamma_\varepsilon} U \frac{\mathrm{d}\phi}{\mathrm{d}r} \mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \phi \frac{\mathrm{d}U}{\mathrm{d}r} \mathrm{d}\Gamma \right\} = -\phi(0) \tag{14}$$

As ϕ is a test function, its derivative is limited, which means to say that[4,5]

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| \leq M \tag{15}$$

where M has a positive and limited value.

Solving eqn (14) for the two-dimensional case, it can be seen that the first integral is zero, while the remainder results in

$$C_1 = \frac{-1}{2\pi} \tag{16}$$

In this way one concludes that the natural logarithm elementary function is an essential function, while the constant elementary function is a complementary function due to the fact that no restrictions are imposed on the value of C_2. Making use of the substitution

$$C_2 = F_2 C_1 \tag{17}$$

the general fundamental solution of Laplace's operator in the two-dimensional domain can be expressed as

$$U(r) = \frac{-1}{2\pi}[\ln(r) + F_2] \tag{18}$$

Similarly, for the three-dimensional case consider eqn (13) and one obtains

$$U(r) = \frac{1}{4\pi}\left(\frac{1}{r} + F_2\right) \tag{19}$$

As shown above, it can be seen that Brebbia *et al.*[1] have considered the fundamental solution of Laplace's two- and three-dimensional operators disregarding the complementary function associated with F_2.

A helpful fact on the numerical implementation of logarithmic functions like eqn (18) is that, due to the arbitrary coefficient F_2, this equation can be written as

$$U(r) = \frac{-1}{2\pi}\ln(\alpha r) \tag{20}$$

where $\alpha = \ln(F_2)$. Now, the coefficient α can be selected in such a way that the problem can be scaled in order to give the natural logarithm argument such that the results neither overflow nor are incalculable. Because F_2 can be any real value, it can be seen that $\alpha > 0$. In this case α represents a scaling in the value of the radius r. This scaling produces no change in the value of any tensor derived from $U(r)$.

3.2 Bi-harmonic operator

The representative equation of Kirchoff's plate model is expressed as[8]

$$-D\Delta^2 u = -b \tag{21}$$

and is written in this way in order to be in accordance with eqns (2) and (3). Here, $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity, E is the modulus of elasticity, ν is the Poisson coefficient, h is the plate thickness and $\Delta^2 = \Delta\Delta$ is the bi-harmonic operator.

As in the case of the Laplace eqn (11) this equation can be written as follows

$$-D\Delta^2 U(r) = -\delta(r) \tag{22}$$

whose homogeneous two-dimensional general solution is

$$U(r) = C_1 r^2 \ln(r) + C_2 r^2 + C_3 \ln(r) + C_4 \tag{23}$$

Following the same procedure as in Laplace's operator, and applying Green's formula successively to eqn (7), there results

$$\lim_{\varepsilon \to 0}\left[\int_{\Gamma_\varepsilon} U\Delta\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\mathrm{d}\Gamma - \int_{\Gamma_\varepsilon} \frac{\mathrm{d}U}{\mathrm{d}r}\Delta\phi \mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \Delta U \frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}\Gamma - \int_{\Gamma_\varepsilon} \phi\Delta\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)\mathrm{d}\Gamma\right] = \frac{-\phi(0)}{D} \tag{24}$$

Bearing in mind that the derivatives of the test functions are limited, one can see that the first and the third integrals are zero. To regularize the second integral it is necessary that

$$C_3 = 0 \tag{25}$$

while, from the last integral, satisfying the equality, one can conclude that

$$C_1 = \frac{1}{8\pi D} \tag{26}$$

By the substitutions

$$C_2 = F_2 C_1$$
$$C_4 = F_4 C_1 \tag{27}$$

it results that the general fundamental solution for the Kirchhoff's plate model is

$$U(r) = \frac{1}{8\pi D}[r^2 \ln(r) + F_2 r^2 + F_4] \tag{28}$$

Only the essential part of this solution ($F_2 = F_4 = 0$) is presented by Brebbia *et al.*,[1] while Costa and Brebbia[9] concluded that $F_2 = -1/2$ is a good value. Brebbia and Dominguez,[2] applying Laplace's operator to the above fundamental solution, with $F_2 = +1$ and $F_4 = 0$, obtained the fundamental solution of Laplace's two-dimensional operator (eqn (18)) in its essential form ($F_2 = 0$). Taking the F_2 value as proposed by Costa and Brebbia[9] one obtains $F_2 = -3/2$ in eqn (18).

In the same way as for eqn (20), one can write

$$U(r) = \frac{1}{8\pi D}[r^2 \ln(\alpha r) + F_4] \tag{29}$$

3.3 Reissner/Mindlin's plate models

The operator applied to the plate models of Reissner[10] and Mindlin[11] is expressed exactly in the form presented in eqn (2), with the differential operators given by[4,12,13]

$$\mathbf{L} = D\frac{(1-\nu)}{2}\begin{bmatrix} \left[\Delta - \lambda^2 + \frac{(1+\nu)\partial^2}{(1-\nu)\partial x_1^2}\right] & \frac{(1+\nu)}{(1-\nu)}\frac{\partial^2}{\partial x_1 \partial x_2} & -\lambda^2 \frac{\partial}{\partial x_1} \\ \frac{(1+\nu)}{(1-\nu)}\frac{\partial^2}{\partial x_1 \partial x_2} & \left[\Delta - \lambda^2 + \frac{(1+\nu)}{(1-\nu)}\frac{\partial^2}{\partial x_2^2}\right] & -\lambda^2 \frac{\partial}{\partial x_2} \\ \lambda^2 \frac{\partial}{\partial x_1} & \lambda^2 \frac{\partial}{\partial x_2} & \lambda^2 \Delta \end{bmatrix} \tag{30}$$

and

$$\mathbf{F} = \begin{bmatrix} MF\frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & MF\frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

where

$$\lambda^2 = 12k^2/h^2 \tag{32}$$

$$k^2 = 5/6 \tag{33}$$

and

$$MF = \begin{cases} \frac{\nu}{(1-\nu)\lambda^2} & \text{, Reissner} \\ 0 & \text{, Mindlin} \end{cases} \tag{34}$$

According to Mindlin's plate model, the shear correction factor k^2 is a function of the Poisson coefficient.

Applying Hörmander's method, eqn (6) is expressed as[13]

$$D^3\left(\frac{1-\nu}{2}\right)\lambda^2\Delta^2(\Delta - \lambda^2)G(r) = -\delta(r) \tag{35}$$

whose homogeneous general solution is

$$G(z) = C_1K_0(z) + C_2\ln(z) + C_3z^2 + C_4z^2\ln(z) + C_5I_0(z) + C_6 \tag{36}$$

where $I_0(z)$ and $K_0(z)$ are the modified Bessel's functions[14] of first and second kind, respectively, and of zero order, and

$$z = \lambda r \tag{37}$$

Applying Green's formula successively to eqn (7), substituting previously the operator presented in eqn (35), one obtains[4]

$$\lim_{\varepsilon\to 0}\left\{ -\int_{\Gamma_\varepsilon} G\Delta(\Delta - \lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon}\frac{\mathrm{d}G}{\mathrm{d}r}\Delta(\Delta-\lambda^2)\phi\mathrm{d}\Gamma - \int_{\Gamma_\varepsilon}\Delta G(\Delta-\lambda^2)\frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon}\Delta\frac{\mathrm{d}G}{\mathrm{d}r}(\Delta-\lambda^2)\phi\mathrm{d}\Gamma - \int_{\Gamma_\varepsilon}\Delta^2G\frac{\mathrm{d}\phi}{\mathrm{d}r}\mathrm{d}\Gamma + \int_{\Gamma_\varepsilon}\phi\Delta^2\frac{\mathrm{d}G}{\mathrm{d}r}\mathrm{d}\Gamma\right\} = -\frac{4}{D^3(1-\nu)^2\lambda^2}\phi(0) \tag{38}$$

The first, third and fifth integrals above are zero. Taking the sixth integral, satisfying the equality, one concludes that

$$C_1 = \frac{2}{\pi D^3(1-\nu)^2\lambda^6} \tag{39}$$

In order for the remaining integrals (the second and the fourth) to be regularly evaluated, it is necessary that

$$C_4 = \frac{1}{4}C_1$$

$$C_2 = C_1 \tag{40}$$

respectively. Moreover, in order to satisfy the regularity condition at infinity, it is necessary that

$$C_5 = 0 \tag{41}$$

The functions related to C_3 and C_6 are the complementary ones of the fundamental solution and therefore, writing

$$C_3 = F_3C_1$$

$$C_6 = F_6C_1 \tag{42}$$

the general scalar fundamental solution is expressed as

$$G(z) = \frac{2}{\pi D^3(1-\nu)^2\lambda^6}\left[K_0(z) + \ln(z) + F_3z^2 + \frac{1}{4}z^2\ln(z) + F_6\right] \tag{43}$$

Substituting this equation in (4) the general tensor

$U_{ij}(r)$ is obtained as

$$U_{\alpha\beta}(z) = \frac{1}{8\pi D(1-\nu)}\{[8B(z) - (1-\nu)(2\ln(z) + 1 + 8F_3)]\delta_{\alpha\beta} - [8A(z) + 2(1-\nu)]r_{,\alpha}r_{,\beta}\}$$

$$U_{\alpha 3}(z) = \frac{1}{8\pi D}[2\ln(z) + 1 + 8F_3]rr_{,\alpha}$$

$$U_{3\alpha}(z) = -U_{\alpha 3}(z)$$

$$U_{33}(z) = \frac{1}{8\pi D(1-\nu)\lambda^2}\{(1-\nu)z^2(\ln(z) + 4F_3) - 8\ln(z) - 4[(3-\nu)(4F_3+1) - (1-\nu)F_6]\} \tag{44}$$

where

$$A(z) = K_0(z) + \frac{2}{z}\left(K_1(z) - \frac{1}{z}\right)$$
$$B(z) = K_0(z) + \frac{1}{z}\left(K_1(z) - \frac{1}{z}\right) \tag{45}$$

A compact form of the above equations can be obtained by defining α and β such that

$$\ln(\alpha) = (1 + 8F_3)/2$$
$$\ln(\beta) = [(3-\nu)(4F_3+1) - (1-\nu)F_6]/2 \tag{46}$$

resulting in

$$U_{\alpha\beta}(z) = \frac{1}{4\pi D(1-\nu)}\{[4B(z) - (1-\nu)\ln(\alpha z)]\delta_{\alpha\beta} - [4A(z) + (1-\nu)]r_{,\alpha}r_{,\beta}\}$$

$$U_{\alpha 3}(z) = \frac{1}{4\pi D}\ln(\alpha z)rr_{,\alpha}$$

$$U_{3\alpha}(z) = -U_{\alpha 3}(z)$$

$$U_{33}(z) = \frac{1}{8\pi D(1-\nu)\lambda^2}\left[(1-\nu)z^2\left(\ln(\alpha z) - \frac{1}{2}\right) - 8\ln(\beta z)\right] \tag{47}$$

The expression for $U_{ij}(r)$ above is both slightly different and more compact than those presented by Weeën[13] and, consequently, more adequate for numerical implementation. The tensor $U_{ij}(r)$ is reduced to that presented previously by Barcellos and Westphal[15] if $\alpha = \beta = 1$, but here the fundamental solution is also complete. As seen before, α and β have any values greater than zero.

The tensor $T_{ij}(r)$ is shown in the Appendix in order for the study to be complete. The remaining general tensors are given by Barcellos and Westphal[15] in terms of F_3 and F_6.

3.4 Two-dimensional elasticity

Navier's equations for two-dimensional elasticity are expressed as[1]

$$\mu\left(u_{\alpha,\beta\beta} + \frac{1}{(1-2\nu)}u_{\beta,\beta\alpha}\right) = -b_\alpha \tag{48}$$

where μ is the shear modulus.

Therefore, the differential operator $L_{\alpha\beta}$ is

$$\mathbf{L} = \mu\begin{bmatrix} \Delta + \dfrac{1}{1-2\nu}\dfrac{\partial^2}{\partial x_1^2} & \dfrac{1}{1-2\nu}\dfrac{\partial^2}{\partial x_1 \partial x_2} \\ \dfrac{1}{1-2\nu}\dfrac{\partial^2}{\partial x_1 \partial x_2} & \Delta + \dfrac{1}{1-2\nu}\dfrac{\partial^2}{x_2^2} \end{bmatrix} \tag{49}$$

For this case eqn (6) is

$$\frac{2(1-\nu)}{(1-2\nu)}\mu^2\Delta^2 G(r) = -\delta(r) \tag{50}$$

Following the steps performed to obtain eqn (28), it is found that

$$G(r) = \frac{-(1-2\nu)}{16\pi(1-\nu)\mu^2}[r^2\ln(r) + F_2 r^2 + F_4] \tag{51}$$

Consequently, the tensor $U_{\alpha\beta}(r)$ is obtained through eqn (4) as

$$U_{\alpha\beta}(r) = \frac{-1}{8\pi(1-\nu)\mu}\left\{\left[(3-4\nu)\ln(r) + (3-4\nu)F_2 + \frac{(7-8\nu)}{2}\right]\delta_{\alpha\beta} - r_{,\alpha}r_{,\beta}\right\} \tag{52}$$

When $F_2 = 0$, this is exactly the same equation as that obtained by Galerkin's vector.[2] The term $[(3-4\nu)F_2 + (7-8\nu)/2]\delta_{\alpha\beta}$ is related to a rigid body translation in the Cartesian directions 1 and 2. In this way, eqn (52) is similar to that suggested by Telles and de Paula,[16] Neves and Brebbia,[17] Tomás Pereira[18] and Kuhn *et al.*[19,20] Telles and de Paula[16] used this term to satisfy the equilibrium of forces and moments starting from the basic equations of BEM. Neves and Brebbia[17] and Tomás Pereira,[18] by using numerical experiments, showed that the final solution of a problem depends on the adequate choice of the value of this term. Kuhn *et al.*[19,20] concluded that there are two critical values of the rigid body motions in directions 1 and 2, where the matrix to be solved is singular.

A compact form of the above equations can be obtained, as before, in the form

$$U_{\alpha\beta}(r) = \frac{-1}{8\pi(1-\nu)\mu}[(3-4\nu)\ln(\alpha r)\delta_{\alpha\beta} - r_{,\alpha r,\beta}] \tag{53}$$

3.5 Three-dimensional elasticity

Navier's equations for three-dimensional elasticity are analogous to eqns (48), with the Greek indices changed to Latin indices. Equation (50) is, for this case

$$\frac{2(1-\nu)}{(1-2\nu)}\mu^3\Delta^3 G(r) = -\delta(r) \tag{54}$$

The homogeneous solution is

$$G(r) = C_1 r^4 + C_2 r^3 + C_3 r^2 + C_4 r + C_5 \frac{1}{r} + C_6 \quad (55)$$

The two equations above, when substituted into eqn (7), and after applying Green's formula successively, results in

$$\lim_{\varepsilon \to 0} \left\{ -\int_{\Gamma_\varepsilon} G \frac{\mathrm{d}(\Delta^2 \phi)}{\mathrm{d}r} \mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \frac{\mathrm{d}G}{\mathrm{d}r} \Delta^2 \phi \mathrm{d}\Gamma - \int_{\Gamma_\varepsilon} \Delta G \frac{\mathrm{d}(\Delta\phi)}{\mathrm{d}r} \mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \frac{\mathrm{d}(\Delta G)}{\mathrm{d}r} \Delta\phi \mathrm{d}\Gamma - \int_{\Gamma_\varepsilon} \Delta^2 G \frac{\mathrm{d}\phi}{\mathrm{d}r} \mathrm{d}\Gamma + \int_{\Gamma_\varepsilon} \frac{\mathrm{d}(\Delta^2 G)}{\mathrm{d}r} \phi \mathrm{d}\Gamma \right\} = \frac{-(1-2\nu)}{2(1-\nu)\mu^3} \phi(0) \quad (56)$$

The first, third and fifth integrals above are zero. Taking the sixth integral, satisfying the equality, it follows that

$$C_2 = \frac{(1-2\nu)}{192\pi(1-\nu)\mu^3} \quad (57)$$

For the remaining integrals (the second and fourth), one has to define

$$C_4 = 0 \qquad C_5 = 0 \quad (58)$$

in order for these integrals to be regular.

Then, by writing

$$C_1 = F_1 C_2 \qquad C_3 = F_3 C_2 \qquad C_6 = F_6 C_2 \quad (59)$$

the fundamental solution $G(r)$ can be rewritten as

$$G(r) = \frac{(1-2\nu)}{192\pi(1-\nu)\mu^3} [F_1 r^4 + r^3 + F_3 r^2 + F_6] \quad (60)$$

Finally, the fundamental tensor $U_{ij}(r)$ is obtained through eqn (4) as

$$U_{ij}(r) = \frac{1}{16\pi(1-\nu)\mu} \left\{ \left[\frac{(3-4\nu)}{r} + \frac{10}{3}(5-6\nu)F_2 \right] \delta_{ij} + \frac{1}{r} r_{,i} r_{,j} \right\} \quad (61)$$

or, by making an adequate substitution,

$$U_{ij}(r) = \frac{1}{16\pi(1-\nu)\mu} \left\{ (3-4\nu)\left(\frac{1}{r} + F\right)\delta_{ij} + \frac{1}{r} r_{,i} r_{,j} \right\} \quad (62)$$

where F is a constant.

As in two-dimensional elasticity, there are here three rigid body motions incorporated in the fundamental solution, related to three critical values of the F constant for each particular problem.

4 CONCLUSIONS

In this paper it was shown that general fundamental solutions of elliptic linear differential equations with constant coefficients can be easily obtained by using Hörmander's method, in conjunction with the theory of distributions.

The rigid body motions are consistently incorporated into fundamental solutions, when the general scalar differential equation solution is carried out in full through the derivation of the tensor $U_{ij}(r)$.

Computational savings and improvements can be attained with skillful numerical implementation of general fundamental solutions.

ACKNOWLEDGEMENT

The authors are indebted to the National Council for Scientific and Technological Development (CNPq), Brazil, for the financial support of this work.

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APPENDIX: THE GENERAL FUNDAMENTAL TENSOR $T_{ij}(r)$ OF REISSNER/MINDLIN'S PLATE MODELS

The general fundamental tensor $T_{ij}(r)$ of Reissner/Mindlin's plate models, as obtained starting from the general fundamental displacement tensor $U_{ij}(r)$ (eqn (47)), is expressed as:

$$T_{\alpha\beta}(z) = \frac{-1}{4\pi r}[(4A(z) + 2zK_1(z) + 1 - \nu)$$

$$(r_{,\beta}n_\alpha + r_{,n}\delta_{\alpha\beta}) + (4A(z) + 1 + \nu)r_{,\alpha}n_\beta$$

$$- 2(8A(z) + 2zK_1(z) + 1 - \nu)r_{,\alpha}r_{,\beta}r_{n}]$$

$$T_{\alpha 3}(z) = \frac{\lambda^2}{2\pi}[B(z)n_\alpha - A(z)r_{,\alpha}r_{,n}]$$

$$T_{3\alpha}(z) = \frac{-1}{4\pi}\{[(1+\nu)\ln(\alpha z) + \nu]n_\alpha + (1-\nu)r_{,\alpha}r_{,n}\}$$

$$T_{33}(z) = \frac{-1}{2\pi r}r_{,n} \tag{A1}$$

where $r_{,n} = r_{,\alpha}n_\alpha$ and n_α are the direction cosines of the outward normal to the boundary.