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Investigations on the hp-Cloud Method by solving Timoshenko beam problems

P. de T. R. Mendonça, C. S. de Barcellos, A. Duarte

Abstract The hp-Cloud Method is a new and promising approximation technique which, without relying on a mesh, can be used in both finite and boundary element methods. In spite of its great success in solving problems with high accuracy and convergence rates, there are still a number of aspects to be qualitative and quantitatively investigated. Among these are: the sensitivity to the weight functions used in the construction of the Shepard functions, the sensitivity to the class of enrichment functions used in the p-adaptivity, the sensitivity to the cloud overlapping, and the variations of the condition number. This paper describes numerical experiments regarding some of the many choices allowed by the hp-Cloud methodology applied to Timoshenko beam problems. Since the Moving Least Squares Method is used to generate the partition of unity, some choices of weighting functions are studied and the results are compared to each other. In addition, convergence results are presented for successive h-refinements, when the number of clouds is increased, and for increasingly higher order approximation functions characterizing p-refinements. Since the new basis functions are in general not polynomials, an adaptive integration procedure is employed. The efficiency of several types of basis functions is verified. The rates of h and p convergence are determined as functions of other parameters. Also, examples of degradation of the stiffness matrix condition number is displayed and discussed.

1 Introduction

Many continuum mechanics problems like crack propagation, large deformation and shape optimization are characterized by a continuous change of the domain and this may require several remeshings. Even when few meshes are needed, this modeling can be quite expensive and time consuming besides being error prone. This has motivated along the last decade a more intensive search for numerical procedures which preclude the use of a mesh for accurately representing the solution shape, although an auxiliary mesh may be needed for integration purposes. Such new concepts have been implemented both in the Finite Element Method and Boundary Element Method. The main proposals which follow the meshless concepts are: Diffuse Element Method, DEM, (Nayroles et al. 1992); Smoothed Particle Hydrodynamics Method, SPH, (Gingold and Monaghan 1982); Element Free Galerkin Method, EFGM, (Belytschko et al. 1993; Belytschko et al. 1994; Krysl and Belytschko 1995) and Boundary Node Method (Mukherjee and Mukherjee 1997); Wavelet Galerkin Methods (Amarantuga et al. 1994); Reproducing Kernel Particle Methods (Liu and Chen 1995; Liu et al. 1996); A meshless local boundary integral equation (LBIE) method (Zhu et al. 1998a; Zhu et al. 1998b); A meshless local Petrov-Galerkin (MLPG) method (Atluri and Zhu 1998a; Atluri and Zhu 1998b); Partition of Unity Finite Element Method, PUFEM, (Babuška and Melenk 1995); and the hp-Cloud Method (Duarte and Oden 1995a; Duarte 1995b; Duarte and Oden 1996a; Duarte and Oden 1996b; Duarte 1996c; Nicolazzi et al. 1997a; Nicolazzi et al. 1997b).

The hp-Cloud Method, which was proposed by Duarte and Oden (Duarte and Oden 1995a), includes in its formalism both the h-enrichment, now interpreted as an increase in the nodal density, and the p-enrichment by hierarchically multiplying the cloud partition of unity functions by selected polynomials. In contrast with other proposals, Duarte and Oden have also laid all the mathematical foundations which support the method and the adaptivity criteria from its beginning (Duarte and Oden 1995a).

In the present paper, the basic concepts of the hp-Cloud Method is reviewed and some of the results obtained up to now are presented. They are concerned with the sensitivity of the formulation to different choices of the weight functions used in the Shepard Method, to different choices of the enrichment functions and to different cloud overlapping, as well as the degradation of the condition number of the final stiffness matrix. The hp-Cloud Method has been deeply investigated for solving potential and linear elasticity problems and, presently, one is extending the investigations for higher order problems. The first steps, which are presented here, are based on the

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Timoshenko beam model which includes the same kind of difficulties as the Mindlin/Reissner plate ones. Among these is the locking problem when solving by conventional finite element procedures.

This paper is organized as follows. In Sect. 2, the concepts related to the partition of unity are presented. In Sect. 3, the procedure to approximate the solution is presented and in Sect. 4 it is explained how to improve the approximations by adding new enriched functions. In Sect. 5, the selected variational principles are outlined together with numerical results regarding weight function effects over p-refinement convergence rates, locking effects under h and p refinements, and effects of cloud overlapping. The adopted numerical integration is briefly explained in Sect. 6. Finally, some conclusions are given in Sect. 7.

2
Partition of unity

The FEM-hp-Clouds is a method which is applicable to arbitrary domains and makes use of a randomly distributed set of nodes for describing the approximated solution. Based on error estimators and indicators, the accuracy is increased by appropriate h, or p, or both enrichments. In order to build the approximation functions, the Shepard Method (Lancaster and Salkauskas 1981) is used with radial functions with variable support as weight functions.

In order to introduce the hp-Cloud approximation functions, one initially considers an arbitrary set Q_N of N points $x_\alpha \in \bar{\Omega}$, where $\bar{\Omega}$ denotes the closure of the domain $\Omega \subset R^n$, $n = 1, 2$ or 3, that is

$$Q_N = \{x_1, x_2, \ldots, x_N\}, \quad x_\alpha \in \bar{\Omega} \ . \tag{1}$$

Around each point x_α, a neighborhood ω_α is associated such that the union of all such neighborhoods covers the domain $\bar{\Omega}$. Now one can define Clouds and partition of unity.

Clouds are the elements ω_α of an open covering F_N of the domain. For each cloud there is an associated point x_α.

Partition of Unity subordinated to the open covering F_N is the class of functions $L_N = \{\varphi_\alpha(x)\}$ such that $\sum_\alpha \varphi_\alpha(x) \equiv 1$ for every $x \in \bar{\Omega}$.

There are some additional and more restrictive definitions of partition of unity but all it is needed here is the requirement that the sum of the functions $\varphi_\alpha(x)$ be equal to one at every point $x \subset \bar{\Omega}$. In order to obtain an admissible set L_N, a weight radial function $W_\alpha(x)$ is defined for each cloud ω_α and has this cloud as its compact support.

Hence, the functions $\varphi_\alpha(x)$ are obtained by using the Shepard functions which are simply computed as

$$\varphi_\alpha(x) = \frac{W_\alpha(x)}{\sum_{\beta=1}^{N} W_\beta(x)} \ . \tag{2}$$

It is apparent that the shape of the partition of unity approximations ought to be dependent on each choice of the weight functions and it is presently reported the use of four of them. The essential criteria is to attribute a greater weight to the nearest points x_α of x than those located at farther distance and to have the global continuity required by the problem. This is here accomplished by using decreasing weight functions which are restricted to be nonzero only over its support, here referred to as cloud. The reason for this approach is to obtain local representations similar to the global approximation functions of the Finite Element Method, resulting therefore in sparse coefficient matrices.

As examples of one-dimensional weight functions, one can mention: the tent function built by two straight segments which is similar to those used by Babuška and Melenk (1997) in the PUFEM Method; splines which are zero together with its first "p" derivatives at the cloud boundary; singular functions in terms, e.g., of $(1/|x - x_\alpha|)^q$ multiplied by functions which are also zero together with its first "p" normal derivatives at the cloud boundary; trigonometric functions; and so on.

3
Approximation in the cloud

Consider a weight function, $W_\alpha(x) \geq 0, \forall x \in \bar{\Omega}$, with $W_\alpha(x) \in C_0^l(\omega_\alpha)$, that is, $W_\alpha(x)$ lies in the space of continuous functions together with its derivatives up to the order l and, in addition, has the cloud ω_α as its compact support. The clouds are here defined as balls, B, with radius h and centered at the points $x_\alpha \in Q_N$. Hence, the weight function associated with the αth cloud can be written as $W_\alpha(x) = W(x - x_\alpha)$ and the associated partition of unity function is evaluated as

$$\varphi_\alpha(x) = \frac{W_\alpha(x)}{\sum_{\beta=1}^{N} W_\beta(x)} \ . \tag{3}$$

By defining a functional over the space $C_0^l(\bar{\Omega})$ as

$$(f, g)_x = \sum_{\alpha=1}^{N} W_\alpha(x) f(x_\alpha) g(x_\alpha) \tag{4}$$

one can easily show that the partition of unity L_N allows the approximation of a continuous function u: $\bar{\Omega} \longrightarrow R$, with $\bar{\Omega} \subset R^n$, $n = 1$, 2 or 3 by

$$(Lu)(x) = \sum_{\alpha=1}^{N} \varphi_\alpha(x) u_\alpha \tag{5}$$

where $u_\alpha = u(x_\alpha)$.

Notice from the relation (5) that the functions of the partition of unity plays the role of the approximation functions used in Finite and Boundary Element methods, except that they do not always have an explicit form and have to be numerically evaluated at each integration point. Moreover, these functions do not have the selective property $\varphi_\alpha(x_\beta) \neq \delta_{\alpha\beta}$ if more than one cloud covers any of the Q_N points or singular weight functions are not used (Duarte 1995b; Lancaster and Salkauskas 1981). Therefore, as in the Least Square Method, the value of the approximation Lu at the point x_α is in general not equal to u_α as usually occurs in FEM and BEM. In addition, note that the covering of an arbitrary integration point is unknown beforehand, so, one must first to identify the covering clouds of a given integration point and then to compute the partition of unity functions as well as its derivatives at that point.

It has been proved (Duarte and Oden 1996b) that the best choice is to use the Shepard functions, which are inexpensive to compute and meet the proposition stated in the previous section, and to p-enrich this set by multiplying them by linearly independent functions. If one denotes the enrichment set of M functions by $\{L_i(x), i = 1, \ldots, M\}$, the full set of approximation functions is

$$\mathscr{F}_N = \left\{ \begin{array}{cccc} \varphi_1 L_1 & \varphi_2 L_1 & \cdots & \varphi_N L_1 \\ \varphi_1 L_2 & \varphi_2 L_2 & \cdots & \varphi_N L_2 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1 L_M & \varphi_2 L_M & \cdots & \varphi_N L_M \end{array} \right\} \tag{6}$$

Equation (6) illustrates the special case of uniform enrichment, but the procedure allows to enrich each cloud ω_α independently from one another. That is, the number and the kind of enrichment functions may vary from cloud to cloud.

Another great advantage is that these functions L_i can be almost anything, that is, they be can polynomials, trigonometric functions, singular functions, Trefftz functions, and so on. The more representative of the physical problem they are the greater is the convergence rate. Indeed, very high convergence rates have been achieved when a wise selection of the enrichment scheme is used, specially if one knows some information about the solution, like, e.g., order of singularity, directions of anisotropy, among others.

4 Weight and enrichment functions

Some aspects of the numerical implementation of the hp-Cloud method to Timoshenko beam problems are investigated with the intent to better understand the behavior of this procedure in more complex multi-dimensional boundary value problems. So, one considers hereafter one dimensional approximations applicable to the problems under consideration. Initially, one restricts ourselves to the situation where every point $x \notin Q_N$ is covered by only two clouds. Since the only requirement is that the set of Clouds covers $\bar{\Omega}$, every point x must belong to at least one cloud. But notice that if a region is covered by just one cloud, the partition of unity will consist of a single constant function over such a region, as shown in Fig. 1. When this occurs, the gradients of this approximating function are zero and do not contribute to the potential energy functional and, in some cases, may lead to almost singular stiffness matrices. These effects are further discussed in Sect. 5. Next, one considers only the cases in which every point is covered by two Clouds.

Among the several functions which could be chosen, four particular ones are reported in this paper. As a matter of illustration, consider a sequence of four points with coordinates: x_0, x_1, x_2, and x_3 such that $x_0 < x_1 < x_2 < x_3$. Therefore, these points define intervals of length

$$h_0 = x_1 - x_0, \quad h_1 = x_2 - x_1, \quad h_2 = x_3 - x_2 \ . \tag{7}$$

The partitions of the unity which cover a point x in the interval x_1–x_2 are defined by two weight functions, $w_1(x)$ and $w_2(x)$. The considered weight functions are the following:

(i) Tent function

$$w_1(x) = \begin{cases} \frac{x-x_0}{h_0} & x \in [x_0, x_1] \\ \frac{x_2-x}{h_1} & x \in [x_1, x_2] \end{cases} \tag{8}$$

(ii) LN-1 function

$$w_1(x) = \left[1 + \frac{x - x_1}{h_0}\right]^4 \left[1 - \frac{x - x_1}{h_1}\right]^4 \quad x \in [x_0, x_1] \tag{9}$$

(iii) LN-2 function

$$w_1(x) = \left[1 + \frac{x - x_1}{h_0}\right]^2 \left[1 - \frac{x - x_1}{h_1}\right]^2 \quad x \in [x_0, x_1] \tag{10}$$

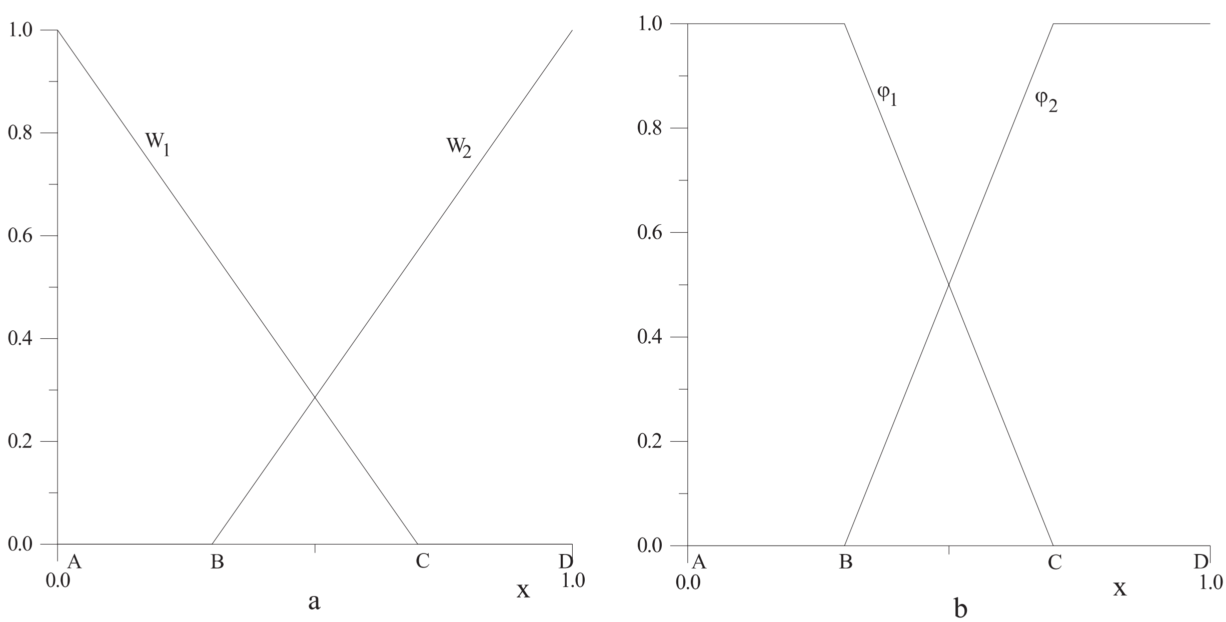

Fig. 1a, b. Model with two nodes, A and D, with cloud overlapping; **a** Weight functions; **b** Partition of unity

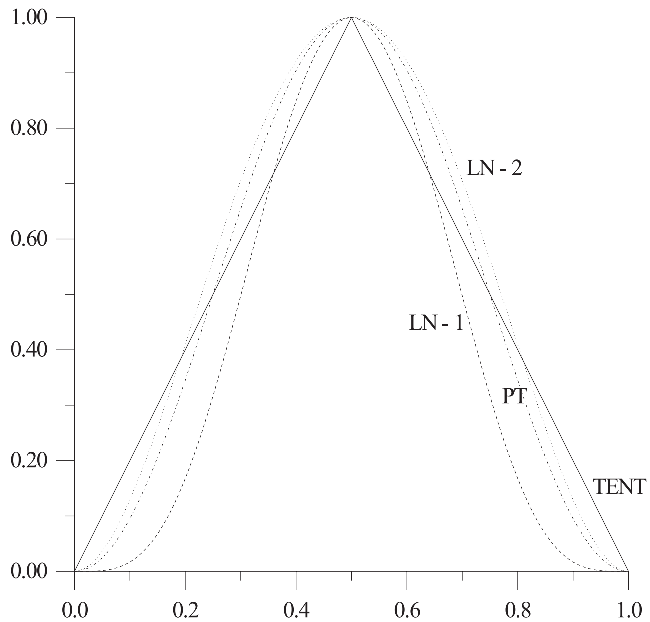

Fig. 2. Weight functions defined over the interval [0, 1]

(iv) PT function

$$w_1(x) = \begin{cases} \cos^2 \frac{\pi(x_1 - x)}{2h_0} & x \in [x_0, x_1] \\ \cos^2 \frac{\pi(x - x_1)}{2h_1} & x \in [x_1, x_2] \end{cases} \tag{11}$$

and the $w_2(x)$ functions are respectively defined by increasingly by one the x and h indices.

The tent function results in an approximation function which is also a piecewise linear function and therefore has continuity $C^0(\Omega)$. The LN-1 function (Nicolazzi et al. 1997a; Nicolazzi et al. 1997b) is a $C^3(\Omega)$ function. The LN-2 function is a variant of the former and is $C^1(\Omega)$, as well as the PT weight function. The tests performed indicate differences in the performance of these functions. Figure 2 qualitatively depicts these four weight functions over the interval $[0, 1]$.

Two basis were chosen as enrichment functions. The first one consists of polynomial functions

$$L_i(x) = x^{i-1} \quad \text{for } i = 1, 2, \ldots, M, \quad x \in \Omega$$

and the second one consists of trigonometric functions

$$L_1(x) = 1$$

$$L_i(x) = \sin \frac{i\pi x}{2l}$$

$$L_{i+1}(x) = \cos \frac{i\pi x}{2l}, \quad i = 2, 4, 6, \ldots, M, \; x \in \Omega$$

The coordinate x is measured as a global coordinate in the same way as in the weight functions, although they could as well be defined in terms of local coordinates. Here l is a representative length of the size of the domain, which defines a wave length for each trigonometric function.

5 Numerical results

Presently, one is concerned with Timoshenko beam model problems because it is a higher order problem when compared to elasticity and potential problems, presents the same sort of locking problems as the Mindlin/Reissner plate models, and brings the advantages of being one dimensional. Consider a cantilever Timoshenko beam, Fig. 3, loaded either by a concentrated load, $F = 1000$ N, at its tip or by a distributed sinusoidal load, $q = 1000 \sin(\pi x / l)$ N/m, over its length, l.

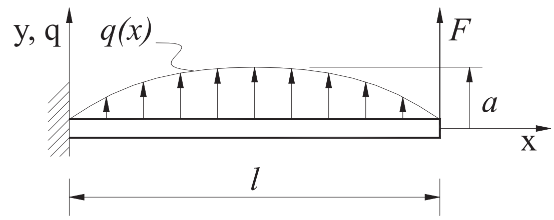

Fig. 3. Timoshenko beam loading

For comparisons with analytical solutions, one considers the Total Potential Energy Functional

$$\Pi = \int_0^l \left\{ \frac{EI}{2} \Psi_{,x}^2 + \frac{GA_s}{2} (v_{,x} + \Psi)^2 - q \cdot v \right\} \mathrm{d}x - F \cdot v(l) \tag{12}$$

where: E and G are the longitudinal and transversal modulus of elasticity, respectively; I and A_s are the respective moment of inertia and shear area of the transversal cross section; and v and Ψ are the linear and angular cross section displacements, respectively, and belong to the space of functions $H^1(\Omega)$.

After applying the Principle of Minimum Potential Energy, which requires that the first variation of the above functional be equal to zero for any kinematic admissible displacement field, one obtains the following differential equations and boundary conditions:

$$-EI \; \Psi_{,xx} + GA_s(v_{,x} + \Psi) = 0 \tag{13}$$

$$-GA_s(v_{,xx} + \Psi_{,x}) = q \tag{14}$$

$$\Psi(0) = 0 \quad M(l) = EI \; \Psi_{,x} = 0 \tag{15}$$

$$v(0) = 0 \quad Q(l) = GA_s(v_{,x} + \Psi) = F$$

In the first load case, only the concentrated tip load is applied and the solution is:

$$v(x) = \frac{F}{2EI} \left(l \cdot x^2 - \frac{x^3}{3} \right) + \frac{F}{GA_s} \cdot x \tag{16}$$

$$\Psi(x) = \frac{Fx}{EI} \cdot \left(\frac{x}{2} - l \right) \tag{17}$$

and, under the distributed load $q(x) = a \cdot \sin(\pi x / l)$, the solution can be written as:

$$v(x) = \frac{a}{EI} \left[\left(\frac{l}{\pi} \right)^4 \cdot \sin \frac{\pi}{x} l - \left(\frac{l}{\pi} \right)^3 \cdot x - \frac{1}{\pi} \left(\frac{x^3}{6} - l \cdot \frac{x^2}{2} \right) \right] + \frac{a}{GA_s} \left[\left(\frac{l}{\pi} \right)^2 \cdot \sin \frac{\pi x}{l} + \frac{l}{\pi} x \right] \tag{18}$$

$$\Psi(x) = \frac{a}{EI}\left[-\left(\frac{l}{\pi}\right)^3\left(\cos\frac{\pi x}{l} - 1\right) + \frac{l}{\pi}\left(\frac{x^2}{2} - l \cdot x\right)\right] \quad (19)$$

Presently, one considers the beam to be $l = 0.45$ m long, except in locking models where one takes $l = 16$ m, and the remaining data are as follows:

$E = 1.97 \cdot 10^{11}$ Pa $A_s = 1.33776 \cdot 10^{-4}\ \mathrm{m}^2$
$G = 7.576923 \cdot 10^{10}$ Pa $I = 8.118207 \cdot 10^{-7}\ \mathrm{m}^4$

Using these data, several models were solved for investigating characteristics of the cloud methodology. In all of them the cloud centers are equally spaced, symmetric, and with equal radii, h. Among the several aspects which deserve consideration, the next four subsections discuss some of the first concerns. Moreover, the convergence results are measured in relative error energy norm, that is

$$E = \sqrt{\frac{U - U_0}{U_0}} \quad (20)$$

where U and U_0 are the strain energies evaluated through the first two terms of Eq. (12) for the approximate and exact solutions, respectively.

5.1
Weight function effect on p-refinement

Here only two clouds centered at the beam ends were used and the number of enrichment functions for each weight function was increased. The relative error energy norm versus the number of polynomial or trigonometric enrichment functions for the four different weight functions defined in Eqs. (8)–(11) are shown in Fig. 4. Here, the analytical solution is reached with the tent weight function and 3 polynomial enrichment functions, because the solution for the problem is a cubic polynomial as indicated in the corresponding curve. The fact that the solution is a polynomial also suggests that the relatively low rates of convergence observed in Fig. 4b is due to the type of enrichment functions used. The trigonometric functions would be more appropriate to problems with harmonic solutions.

The results shown in Fig. 4b for the trigonometric enrichment present, local plateaus, that is, the addition of a new cosine enrichment function sometimes does not strongly improve the approximate solution. This suggests that the solution is almost orthogonal to some of the cosines used in the enrichment set.

Figure 5 show the respective evolution of the condition number, ratio between the greatest and the smallest stiffness matrix eigenvalues, for both cases. The trigonometric enrichment seems to result in rates of variation similar for any weight function. The polynomial functions, in contrast, seems to result in rates of variation of the condition number dependent on the type of the weight functions used.

Comparison between Figs. 4a and 5a indicates that, as the degree of the weight function increases so does the rates of change of the condition number, but the rate of convergence slightly decreases. It can be noted that the function LN-1 is an incomplete polynomial of degree 8, the LN-2 function is of degree 4 and the tent function is of degree 1. The PT function is trigonometric in nature, but does not deviate excessively from a fourth degree Taylor's expansion. Accordingly, its results does not deviate very much from those obtained from the LN-2 function.

5.2
Locking effect under p and h refinements

For this study a long beam, $l = 16$ m, was considered instead. An h-refinement was performed by increasing the number of clouds in the model for some kinds of weight functions and types and number of enrichment functions. The results for the relative error energy norm versus the number of degrees of freedom in the model for different weight functions, and 1 and 3 polynomial

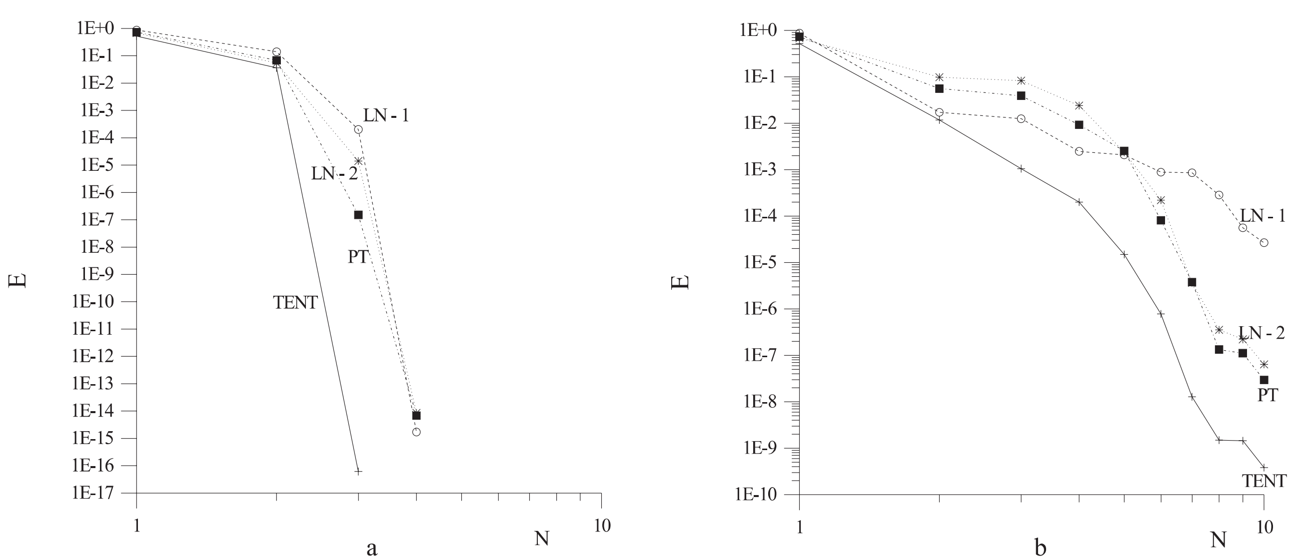

Fig. 4a, b. Relative error energy norm E versus the number N of polynomial **a** and trigonometric **b** enrichment functions for different weight functions and for a two cloud Timoshenko beam model

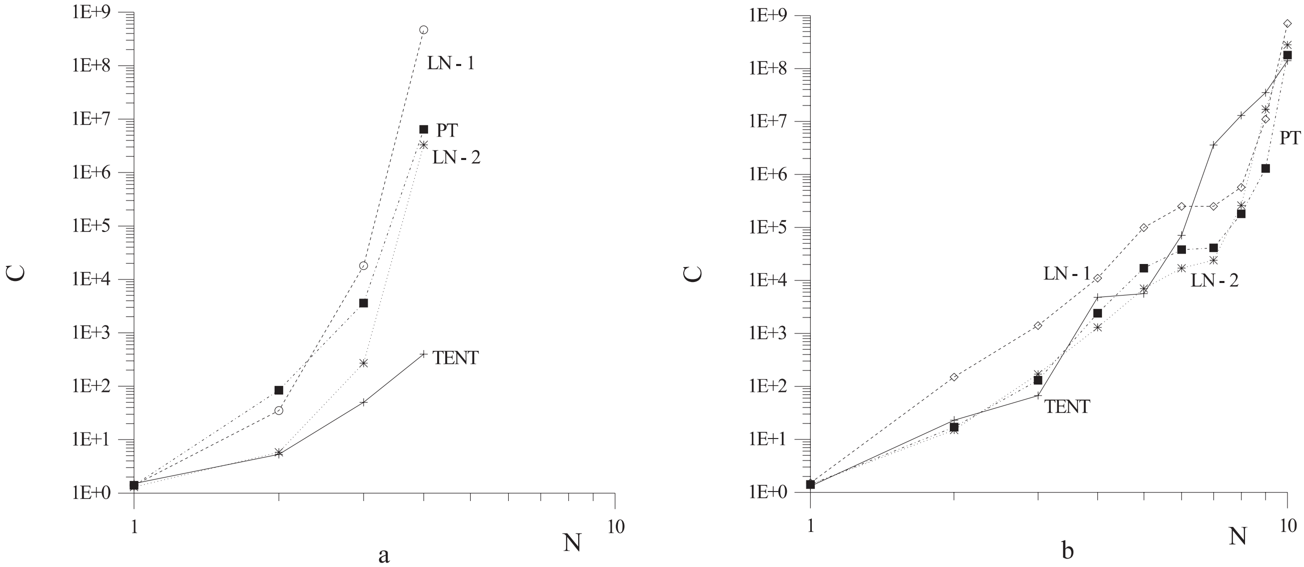

Fig. 5a, b. Condition number C versus number of polynomial **a** and trigonometric **b** enrichment functions for different weight functions and for a two cloud Timoshenko beam model

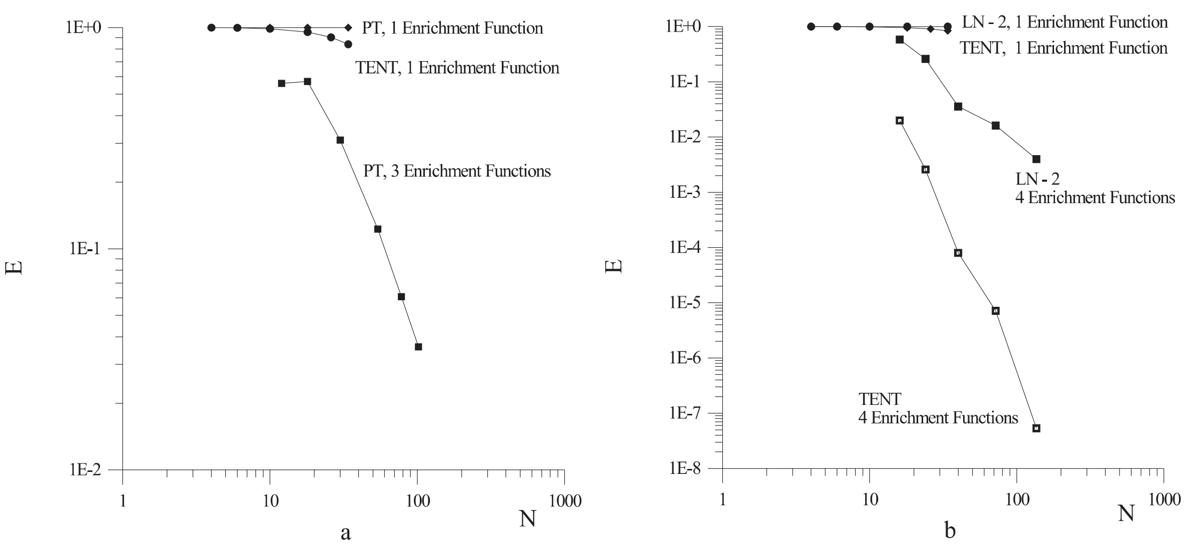

Fig. 6a, b. Relative error energy norm versus the number of degrees of freedom for different weight functions and polynomial **a** and trigonometric **b** enrichment functions for a long Timoshenko beam models under h-refinement

enrichment functions are shown in Fig. 6a. Figure 6b shows the same results when 1 and 4 trigonometric enrichment functions are used. In Figure 6a, the curve for the tent weight function with the polynomial enrichment is not shown. The reason is the following: for each cloud, the weight function is a polynomial of degree one and the enrichment functions. L_i are of degree two. Therefore, the enriched partition of unity is able to represent a complete cubic polynomial and the exact solution can be obtained for the entire beam with a single cloud. Clearly, and a priori knowledge about the nature of the solution of the problem can be used to choose appropriate weight and enrichment functions in order to accelerate the convergence or even to obtain the exact solution. Figure 6 show that the locking effect is present for low order functions and diminishes as the degree of the approximating functions grows. This is indicated in the figures, where the models with only one enrichment function per cloud show nearly no convergence, while for three and four functions the rate of convergence is considerable.

Finally, Fig. 7 show the corresponding evolution of the condition number for both types of enrichment functions versus the number of degrees of freedom. It is observed that the condition number tends to grow exponentially with p-enrichment, while the growth in the h-enrichment happens with constant or decreasing rates. This tendency is in accordance with observations made by Duarte (Duarte 1996c) in connection with plane elasticity and potential problems.

Since one needs here only C^o continuity and the tent weight function showed better results as far as conver-

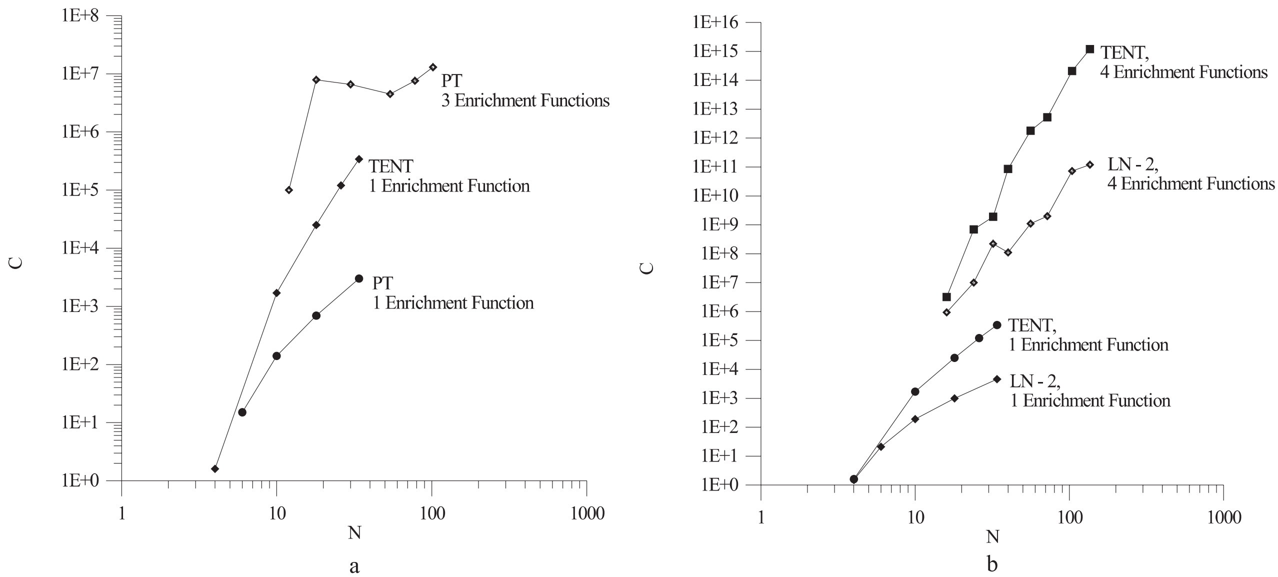

Fig. 7a, b. Condition number C versus the number of degrees of freedom for different weight functions and for the enrichments: **a** 1st and 3rd order polynomial and **b** trigonometric functions

gence rates and condition numbers are concerned, one has restricted the following studies to this sort of weight function which, in addition, has lower computational cost.

5.3
Modified variational principle

Since the partition of unity functions do not have the selective properties due to the arbitrariness of the cloud overlapping, that is $\varphi_j(x_i) \neq \delta_{ij}$, one usually applies the Dirichlet boundary conditions through Lagrange multipliers. In order to avoid increasing the size of the stiffness matrix, these multipliers are identified to the force and bending moment reactions. These are now written in terms of the displacement field variables at the boundary point in the respective Modified Minimum Potential Energy functional which now reads as

$$\begin{aligned}\Pi_M = \int_o^l \left\{ \frac{EI}{2} \Psi_{,x}^2 + \frac{GA_s}{2} (v_{,x} + \Psi)^2 - q \cdot v \right\} \mathrm{d}x \\ - F \cdot v(l) - EI\ \Psi_{,x}(\Psi - \bar{\Psi})|_{x=0} \\ - GA_s(v_{,x} + \Psi)(v - \bar{v})|_{x=0}\end{aligned} \qquad (21)$$

Hence, in the next examples, in which the nodes are covered by more than one cloud, the boundary conditions are only approximately met. In Fig. 8, is shown that the error involved in the boundary condition enforcement also decays as a p-enrichment is effected. The reason for this solution behavior is that now such boundary conditions are not strongly imposed by lines/columns eliminations, but, instead, are applied through stress components which, in addition, have a slower convergence rate than the displacements.

5.4
Effect of cloud overlapping

In this study one considers only the tent weight function with variable coverings due to different, but uniform in

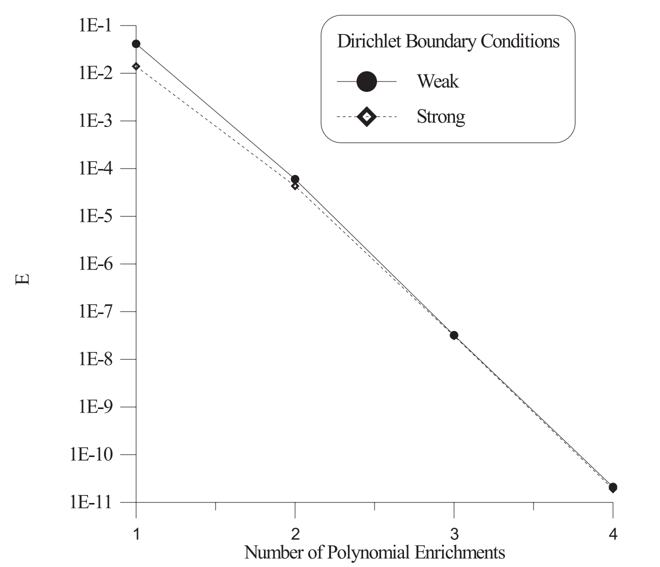

Fig. 8. Comparison of error energy norms for weak and strong Dirichlet boundary conditions

each case, coverage ratios, c. This is defined as the ratio between the clouds radii, h, and the clouds center spacing, l/n_c, where l is the beam length and n_c is the number of clouds minus one. Figure 10a illustrates the case in which the coverage ratio is less than one.

One can immediately verify the deleterious effect when the coverage ratio is less than two in Fig. 10a, where the relative error energy norms for models made of five clouds and enriched with complete fourth order polynomials are depicted for the concentrated load problem. On the other hand, when the coverage ratios c are greater than one the quality also decreases for the same p-refinement, but for the same coverage ratio the results are improved by increasing the polynomial/trigonometric enrichment for the sinusoidal load problem. This can be obviously verified through the Fig. 11 for the concen-

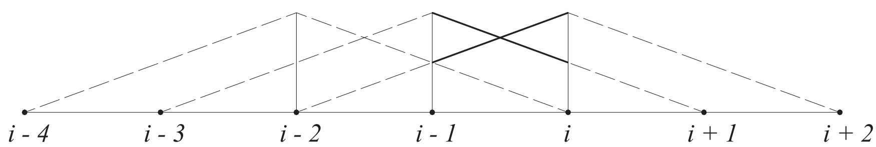

Fig. 9. Clouds overlapping ratio $c = 2$

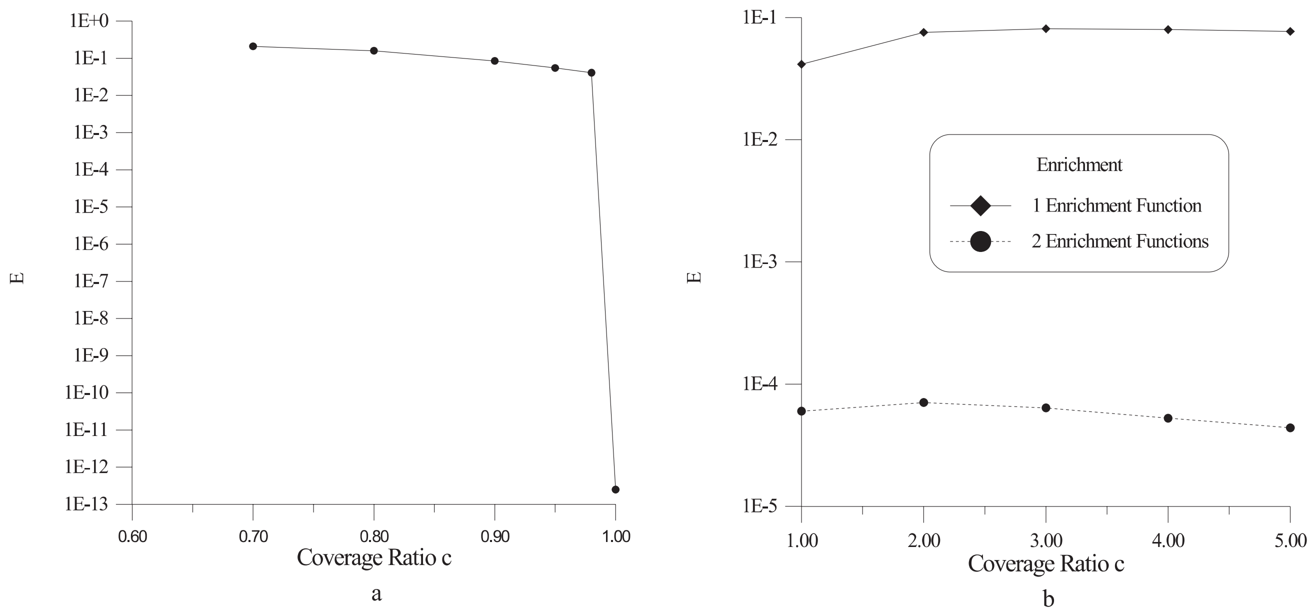

Fig. 10a, b. Loss of accuracy due to clouds coverage ratio: **a** less than one for the concentrated load problem and **b** greater than one for the distributed load problem

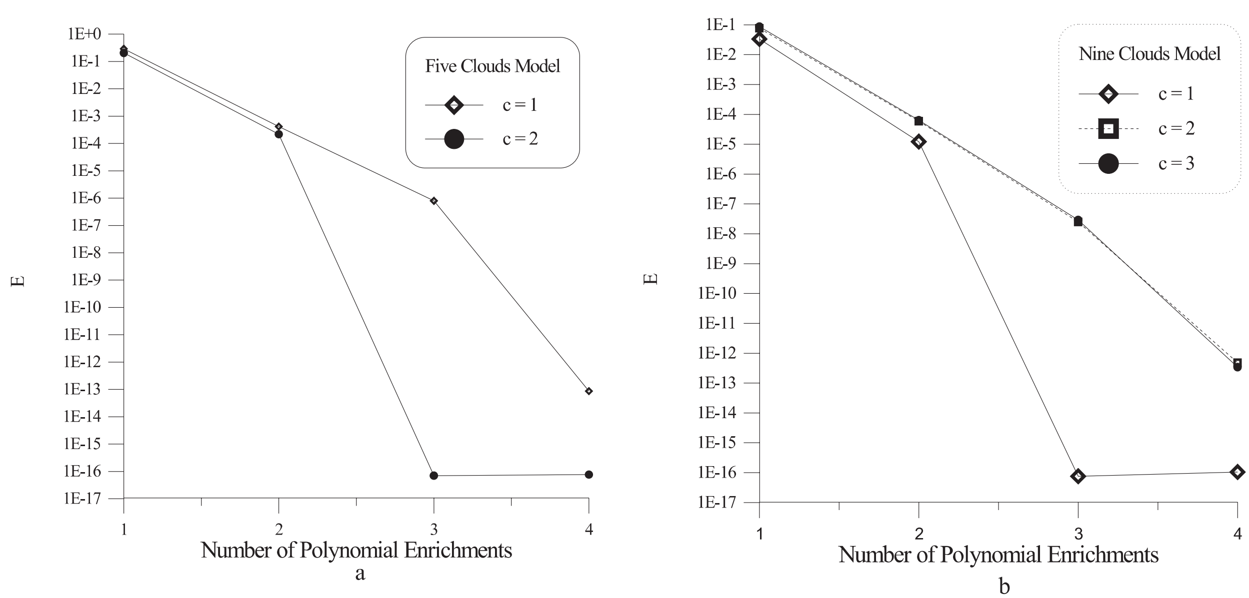

Fig. 11a, b. Loss of accuracy due to coverage ratio greater than one for the concentrated load case

trated load problem modeled by 5 and 9 clouds respectively, and in Fig. 12 for the distributed load. The plateaus which appear in Fig. 11 for $c = 1$ is due to numerical errors since the exact solution is a third degree polynomial. But, one can note from Fig. 10b that for the same p-refinement the error initially increases and then decreases as the coverage ratio increases from one up to five.

Therefore one may conclude that no integration point is to be covered be only one cloud and that the over

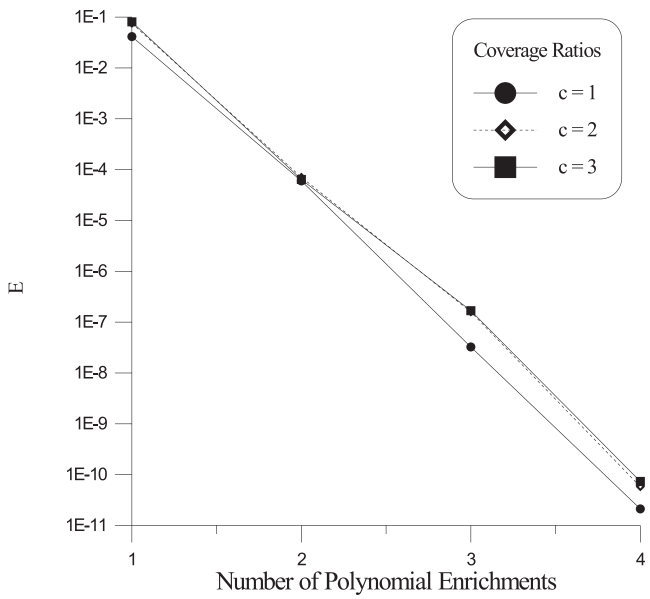

Fig. 12. Loss of accuracy due to coverage ratio greater than one for the distributed load case

coverage, which is inconvenient as far as the stiffness matrix sparsity is concerned, is not as detrimental since the p-enrichment quickly compensates for the error increase.

6 Numerical integration

One of the questions still open, concerns to the most convenient integration scheme to be used. Since the weight functions are generally not polynomials as well as enrichment functions, there is an uncertainty in adopting the most usual integration schemes. Then one has adopted the Petterson scheme for implementing an adaptive procedure (Petterson 1968). This scheme provides a methodology for optimally adding new points to quadratures, that is, if one is using an n point scheme it allows us to determine additional $n+1$ points with optimal coordinates and associated integration weights. Then one has, e.g., the following stages: 1, 3, 7, 15, 31, 63 integration points. One should mention that all the quadrature points are laid inside the domain of integration, e.g., between any two consecutive nodes like x_n and x_{n+1}. For higher dimensional problems, one may use this scheme in the same way as the Gauss-Legendre procedure over an underlying mesh which is used only for integration purposes.

The present procedure consists in integrating by two consecutive stages and compare the relative error to a given tolerance and, if it is still unsatisfactory, one integrates by adding the new points of the next stage and repeating the scheme. Although cumbersome, this procedure avoids neglecting all the previous computations in the next integration stage, which is critical in the Boundary Element Method, and provides confidence in the performed integrations. The aforementioned examples were solved by using up to thirty one integration points.

7 Conclusions

The performed tests confirms that the hp-Cloud FEM presents great versatility in the choice of weight and basis functions. There are indications that the rate of convergence increases as the degree of weight functions is reduced. Weight functions with higher degrees should so be reserved to applications requiring higher smoothness of the approximate solution. These restrictions are not too serious since they concern only to weight functions, and the solution can be better approximated by using higher degree enrichment functions. The growth of the condition number of the coefficient matrix is however of important concern at this point. As far as coverage is concerned, the recommendation is that no integration point should be covered by only one cloud and the over coverage, although detrimental, is preferred since the error quickly decays along the p-enrichment. However, the over coverage has to be avoided due to the high increase in the computational costs due to the partition of unity evaluation and due to a lesser sparsity of the stiffness matrix. In order to avoid under and over integration adaptive schemes are recommended for dealing with arbitrary clouds overlappings and p-enrichments.

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