

Phase Shifting Technique Using Generalization of Carre Algorithm with Many Images

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(Received January 25, 2009; Revised April 24, 2009; Accepted April 24, 2009)

The present work offers new algorithms for phase evaluation in optics measurements. Several phase-shifting algorithms with an arbitrary but constant phase-shift between captured intensity frames are proposed. The algorithms are derived similarly to the so called Carre algorithm. The idea is to develop a generalization of Carre that is not restricted to four images. Errors and random noise in the images cannot be eliminated, but the uncertainty due to its effects can be reduced by increasing the number of observations. The advantages of the proposed algorithm are its precision in the measures taken and immunity to noise in images. © 2009 The Optical Society of Japan

Keywords: phase shifting technique, carre algorithm, phase calculation algorithms, measurement

1. Introduction

Phase shifting is an important technique in engineering.^{1–3} Conventional phase shifting algorithms require phase shift amounts to be known; however, errors on phase shifts are common for the phase shift modulators in real applications, and such errors can further cause substantial errors on the determinations of phase distributions. There are many potential error sources, which may affect the accuracy of the practical measurement, e.g., the phase shifting errors, detector nonlinearities, quantization errors, source stability, vibrations and air turbulence, and so on.⁴

Currently, the phase shifting technique is the most widely used technique for evaluation of interference fields in many areas of science and engineering. The principle of the method is based on the evaluation of the phase values from several phase modulated measurements of the intensity of the interference field. It is necessary to carry out at least three phase shifted intensity measurements in order to determine unambiguously and very accurately the phase at every point of the detector plane. The phase shifting technique offers fully automatic calculation of the phase difference between two coherent wave fields that interfere in the process. There are various phase shifting algorithms for phase calculation that differ on the number of phase steps, on phase shift values between captured intensity frames, and on their sensitivity to the influencing factors during practical measurements.⁵

2. Theory of Phase Shifting Technique

The fringe pattern is assumed to be a sinusoidal function and it is represented by intensity distribution $I(x, y)$. This function can be written in general form as

$$I(x, y) = I_m(x, y) + I_a(x, y) \cos[\phi(x, y) + \delta], \quad (1)$$

where I_m is the background intensity variation, I_a is the

modulation strength, $\phi(x, y)$ is the phase at origin and δ is the phase shift related to the origin.⁶

The general theory of synchronous detection can be applied to discrete sampling procedure, with only a few sample points. At least four signal measurements are needed to determine the phase ϕ and the term δ . Phase Shifting is the preferred technique whenever the external turbulence and mechanical conditions of the images remain constant over the time required to obtain the four phase-shifted frames. Typically, the technique used in this experiment is called the Carre method.⁷ By solving eq. (1) above, the phase ϕ can be determined. The intensity distribution of fringe pattern in a pixel may be represented by the gray level, which varies from 0 to 255. With the Carre method, the phase shift (δ) amount is treated as an unknown value. The method uses four phase-shifted images as

$$\begin{cases} I_1(x, y) = I_m(x, y) + I_a(x, y) \cos[\phi(x, y) - 3\delta/2] \\ I_2(x, y) = I_m(x, y) + I_a(x, y) \cos[\phi(x, y) - \delta/2] \\ I_3(x, y) = I_m(x, y) + I_a(x, y) \cos[\phi(x, y) + \delta/2] \\ I_4(x, y) = I_m(x, y) + I_a(x, y) \cos[\phi(x, y) + 3\delta/2] \end{cases} \quad (2)$$

Assuming the phase shift is linear and does not change during the measurements, the phase at each point is determined as

$$\phi = \arctan \left\{ \frac{\sqrt{[(I_1 - I_4) + (I_2 - I_3)][3(I_2 - I_3) - (I_1 - I_4)]}}{(I_2 + I_3) - (I_1 + I_4)} \right\}. \quad (3)$$

Expanding eq. (3), we obtain the Carre method as

$$\tan(\phi) = \frac{\sqrt{\begin{vmatrix} -I_1^2 & +2I_1I_2 & -2I_1I_3 & +2I_1I_4 \\ & +3I_2^2 & -6I_2I_3 & -2I_2I_4 \\ & & +3I_3^2 & +2I_3I_4 \\ & & & -I_4^2 \end{vmatrix}}}{|-I_1 + I_2 + I_3 - I_4|}. \quad (4)$$

or in a matrix form:

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$$\tan(\phi) = \frac{\sqrt{\left| \sum_{r=1}^4 \sum_{s=r}^4 n_{r,s} I_r I_s \right|}}{\left| \sum_{r=1}^4 d_r I_r \right|} \left\{ \begin{array}{l} \text{Num} = \begin{bmatrix} n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} \\ & n_{2,2} & n_{2,3} & n_{2,4} \\ & & n_{3,3} & n_{3,4} \\ & & & n_{4,4} \end{bmatrix}, \text{Dem} = [d_1 \ d_2 \ d_3 \ d_4] \\ \text{Num} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 3 & -6 & -2 \\ & 3 & 2 \\ & & & -1 \end{bmatrix}, \text{Dem} = [-1 \ 1 \ 1 \ -1] \end{array} \right. \quad (5)$$

Almost all the existing phase-shifting algorithms are based on the assumption that the phase-shift at all pixels of the intensity frame is equal and known. However, it may be very difficult to achieve this in practice. Phase measuring algorithms are more or less sensitive to some types of errors that can occur during measurements with images. The phase-shift value is assumed to be unknown but constant in phase calculation algorithms, which are derived in this article. Consider now the constant but unknown phase shift value δ between recorded images of the intensity of the observed interference field.

Considering N phase shifted intensity measurements, we can write for the intensity distribution I_k at every point of k recorded phase shifted interference patterns

$$I_k(x, y) = I_m(x, y) + I_a(x, y) \cos \left[\phi(x, y) + \left(\frac{2k - N - 1}{2} \right) \delta \right], \quad (6)$$

where $k = 1, \dots, N$ and N being the number of frames.

In Novak,⁴⁾ several five-step phase-shifting algorithms insensitive to phase shift calibration are described, and a

complex error analysis of these phase calculation algorithms is performed. The best five-step algorithm, eq. (7), seems to be a very accurate and stable phase shifting algorithm with the unknown phase step for a wide range of phase step values

$$\begin{cases} a_{jk} = I_j - I_k \\ b_{jk} = I_j + I_k \end{cases}, \quad \tan(\phi) = \frac{\sqrt{4a_{24}^2 - a_{15}^2}}{2I_3 - b_{15}} = \frac{\sqrt{4(I_2 - I_4)^2 - (I_1 - I_5)^2}}{2I_3 - I_1 - I_5}. \quad (7)$$

Expanding eq. (7), we obtain the Novak method as

$$\tan(\phi) = \frac{\sqrt{\begin{bmatrix} -I_1^2 & & & & +2I_1I_5 \\ & +4I_2^2 & & -8I_2I_4 & \\ & & & & +4I_4^2 \\ & & & & & -I_5^2 \end{bmatrix}}}{|-I_1 + 2I_3 - I_5|}, \quad (8)$$

or in matrix of coefficient:

$$\tan(\phi) = \frac{\sqrt{\left| \sum_{r=1}^5 \sum_{s=r}^5 n_{r,s} I_r I_s \right|}}{\left| \sum_{r=1}^5 d_r I_r \right|} \left\{ \begin{array}{l} \text{Num} = \begin{bmatrix} n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} & n_{1,5} \\ & n_{2,2} & n_{2,3} & n_{2,4} & n_{2,5} \\ & & n_{3,3} & n_{3,4} & n_{3,5} \\ & & & n_{4,4} & n_{4,5} \\ & & & & n_{5,5} \end{bmatrix}, \text{Dem} = [d_1 \ d_2 \ d_3 \ d_4 \ d_5] \\ \text{Num} = \begin{bmatrix} -1 & 0 & 0 & 0 & 2 \\ & 4 & 0 & -8 & 0 \\ & & 0 & 0 & 0 \\ & & & 4 & 0 \\ & & & & -1 \end{bmatrix}, \text{Dem} = [-1 \ 0 \ 2 \ 0 \ -1] \end{array} \right. \quad (9)$$

3. Proposed Algorithms

Presently proposed is a general equation for calculating the phase for any number, N , of images $\tan(\phi) = \sqrt{|Num|}/|Dem|$ where

$$\tan(\phi) = \frac{\sqrt{|Num|}}{|Dem|} = \frac{\sqrt{\left| \sum_{r=1}^N \sum_{s=r}^N n_{rs} I_r I_s \right|}}{\left| \sum_{r=1}^N d_r I_r \right|}, \quad (10)$$

or expanding the summations and allowing an arbitrary number of lines

$$\tan(\phi) = \frac{\sqrt{\begin{matrix} n_{1,1}I_1^2 & +n_{1,2}I_1I_2 & +n_{1,3}I_1I_3 & +n_{1,4}I_1I_4 & \cdots & +n_{1,N}I_1I_N \\ & +n_{2,2}I_2^2 & +n_{2,3}I_2I_3 & +n_{2,4}I_2I_4 & \cdots & +n_{2,N}I_2I_N \\ & & +n_{3,3}I_3^2 & +n_{3,4}I_3I_4 & \cdots & +n_{3,N}I_3I_N \\ & & & +n_{4,4}I_4^2 & \cdots & +n_{4,N}I_4I_N \\ & & & & \cdots & \cdots \\ & & & & & +n_{N,N}I_N^2 \end{matrix}}{|d_1I_1 + d_2I_2 + d_3I_3 + d_4I_4 + \cdots + d_{N-1}I_{N-1} + d_NI_N|}}, \tag{11}$$

or emphasizing only the matrix of coefficients of the numerator and the denominator

$$\tan(\phi) = \frac{\sqrt{|Num|}}{|Dem|},$$

$$\left\{ \begin{matrix} Num = \begin{bmatrix} n_{1,1} & n_{1,2} & n_{1,3} & n_{1,4} & \cdots & n_{1,N} \\ & n_{2,2} & n_{2,3} & n_{2,4} & \cdots & n_{2,N} \\ & & n_{3,3} & n_{3,4} & \cdots & n_{3,N} \\ & & & n_{4,4} & \cdots & n_{4,N} \\ & & & & \cdots & \cdots \\ & & & & & n_{N,N} \end{bmatrix} \\ Dem = [d_1 \ d_2 \ d_3 \ d_4 \ \cdots \ d_{N-1} \ d_N] \end{matrix} \right. \tag{12}$$

To display the phase calculation equation in this way permits the viewing of symmetries and plans of a sparse matrix. The use of the absolute value in the numerator and in the denominator restricts the angle between 0 and 90° but avoids negative roots, and, in addition, eliminates the finding

of false angles. Subsequent considerations will later remove this restriction.⁴⁻⁶⁾

In the tested practical applications, an increase of 20% was noticed in the processing time when 16 images were used instead of 4 when processing the standard Carre algorithm, due to many zero coefficients. But if one changes the coefficients from integer type to real, the processing time for the evaluation of phase practically duplicates because real numbers use more memory and more processing time to evaluate floating point additions and multiplications, which are numerous in the equations with a large quantity of images.

The shift on the problem focus of obtaining algorithms for calculating the phase of an analytical problem of a numerical vision is a great innovation and breaks a paradigm that was hitherto used by several authors. After several attempts in numerical modeling the problem, the following mathematical problem was identified

$$\begin{matrix} \text{Minimal} & \sum_{r=1}^N \sum_{s=r}^N |n_{r,s}| + \sum_{r=1}^N |d_r|, \\ \text{subject} & \left\{ \begin{matrix} \text{tan}(\phi) = \text{Sqrt}(|Num|)/|Dem| & \text{number of variables} \\ \text{i) } \tan^2(\phi^v) \left(\sum_{r=1}^N d_r I_r^v \right)^2 = \sum_{r=1}^N \sum_{s=r}^N n_{r,s} I_r^v I_s^v, & v = 1, \dots, \left[\frac{(N+1)N}{2} + N \right] \\ \text{ii) } \sum_{s=r}^N |n_{s,r}| + |d_r| \geq 1, & r = 1, \dots, N, \text{ enter all frames} \\ \text{iii) } \sum_{s=r}^N |n_{r,s}| + |d_r| \geq 1, & r = 1, \dots, N, \text{ enter all frames} \\ \text{iv) } -2N \leq n_{r,s} \leq 2N, & r = 1, \dots, N, s = r, \dots, N \\ \text{v) } -2N \leq d_r \leq 2N, & r = 1, \dots, N \\ \text{vi) } n_{r,s} \text{ are integer,} & r = 1, \dots, N, s = r, \dots, N \\ \text{vii) } d_r \text{ are integer,} & r = 1, \dots, N \end{matrix} \right. \end{matrix}$$

where for each v :

$$\left\{ \begin{matrix} I_k^v(x, y) = I_m^v(x, y) + I_a^v(x, y) \cos \left[\phi^v(x, y) + \left(\frac{2k - N - 1}{2} \right) \delta^v \right], \quad k = 1, \dots, N \\ I_m^v \in [0; 128] \text{ random and real} \\ I_a^v \in [0; 127] \text{ random and real} \\ \phi^v \in [-\pi; \pi] \text{ random and real} \\ \delta^v \in [-2\pi; 2\pi] \text{ random and real} \end{matrix} \right. \tag{13}$$

The coefficients of matrices of the numerator (n_{rs}) and denominator (d_r) must be an integer in order to increase the performance of the computer algorithm, as the values of the intensity of the images (I_k) are also integers ranging from 0 to 255. Modern computers perform integer computations (additions and multiplications) much faster than floating point ones. It should be noted that the commercial digital photographic cameras already present graphics resolution above 12 mega pixels and that the evaluation of phase (ϕ) should be done pixel to pixel. Another motivation is the use of memory: integer values can be stored on a single byte while real values use at least 4 bytes. The present scheme only uses real numbers in the square root of the numerator, the division by denominator, and the arc-tangent over the entire operation.

The idea of obtaining a minimum sum of the values of an absolute or a module of the coefficients of matrices of the numerator (n_{rs}) and denominator (d_r) comes from the attempt to force these factors to zero, for computational speed up and for reducing the required memory, since zero terms in sparse matrices do not need to be stored. It is also important that those ratios are not very large so that the values of the sum of the numerator and of the denominator do not have very high value in order fit into an integer variable. For a precise phase evaluation, these factors will increase the values of the intensity of the images (I_k) that contains errors due to noise in the image, in its discretization in pixels and in shades of gray.

The first restriction of the problem (13) is eq. (10) which is squared to the form of the relation that one is seeking. Note that the results of solving the mathematical problem of the coefficients are matrices on the numerator (n_{rs}) and denominator (d_r), so the number of unknowns is given by ν . To ensure that one has a hyper-restricted problem, the number of restrictions must be greater or at least equal to the number of variables. The ν restrictions of the model are obtained through random choice of values for I_m , I_a , ϕ , and δ and using eq. (6) to compute I_k . Tests showed that for even low numbers for other values of ν , the mathematical problem leads only to one optimal solution, while it becomes more time consuming. Indeed, the values of I_m , I_a , ϕ , and δ can be any real number, but to maintain compatibility with the problem images, it was decided to limit I_m from 0 to 128 and I_a between 0 and 127 so that I_k is between 0 and 255.

The restrictions ii and iii of the problem are based on the idea that all image luminous intensities, I_k , must be present in the equation. This increases the amounts of samples to reduce the noise of random images and requires that all the sampling images enter the algorithm for phase calculation. This is achieved by prescribing that the sum of absolute values of the coefficients of each row or column of the matrix of both the numerator (n_{rs}), plus the module at the rate corresponding to that image in the denominator (d_r) be greater than or equal to 1. Thus the coefficients on the algorithm to calculate the phase for a given image I_k will not be all zeros, ensuring their participation in the equation.

Restrictions iv and v of the problem are used to accelerate the solution of this mathematical model. This limitation in

Table 1. Matrix of coefficient for $N = 4$ and 5, with type $\tan(\phi) = \sqrt{|Num|/|Dem|}$.

N = 4	Num	-1	2	-2	2	
	a)		3	-6	-2	
	Dem	-1	1	1	-1	
N = 5	Num	-1	0	0	0	2
	a)		4	0	-8	0
	Dem	-1	0	2	0	-1

the value of the coefficients of matrices of the numerator (n_{rs}) and denominator (d_r) presents a significant reduction in the universe of search and in the search for a solution of the model optimization. Whenever N is greater than 16, the coefficients of matrices of the numerator (n_{rs}) and denominator (d_r) can be limited to the interval $[-4, 4]$. The search is restricted to coefficients of matrices of the numerator (n_{rs}) and denominator (d_r) which are integers of small value, and since it meets the restrictions of the model, it does not need to be minimized (desirable but not necessary).

Once a solution to the problem is found, it can become a restriction. Therefore, solving the problem again leads to a new and different solution. This allows the problem (13) to lead to many different algorithms for a given value of N , making it very flexible and the numerical problem comprehensive. The following multi-step algorithms for phase calculation use well known trigonometric relations and the branch-and-bound algorithm⁸⁾ for pure integer nonlinear programming with the mathematic problem (13). The following tables show some algorithms (Tables 1, 2, and 3).

Following the model presented of uncertainty analysis in [4, 14], these new equations have excellent results with the application of Monte Carlo-based technique of uncertainty propagation. The Monte Carlo-based technique requires first assigning probability density functions (PDFs) to each input quantity. A computer algorithm is set up to generate an input vector $\mathbf{P} = (p_1, \dots, p_n)^T$; each element p_j of this vector is generated according to the specific PDF assigned to the corresponding quantity p_j . By applying the generated vector \mathbf{P} to the model $Q = M(\mathbf{P})$, the corresponding output value Q can be computed. If the simulating process is repeated n times ($n \gg 1$), the outcome is a series of indications (q_1, \dots, q_n) whose frequency distribution allows us to identify the PDF of Q . Then, irrespective of the form of this PDF, the estimate q_e and its associated standard uncertainty $u(q_e)$ can be calculated by

$$q_e = \frac{1}{n} \sum_{i=1}^n q_i \quad (14)$$

and

$$u(q_e) = \sqrt{\frac{1}{(n-1)} \sum_{i=1}^n (q_i - q_e)^2}. \quad (15)$$

The influence of the error sources affecting the phase values is considered in these models through the values of the intensity I_k . This is done by modifying eq. (6):

Table 2. Matrix of coefficient for $N = 6, \dots, 12$, with type $\tan(\phi) = \sqrt{|Num|}/|Dem|$.

N = 6	Num	-1	0	-1	1	0	2		-1	0	0	0	2		-1	0	-2	2	0	2		-1	2	-2	2	-2	2		-1	2	0	0	-2	2
	Dem	-1	0	1	1	0	-1		-1	0	1	1	0	-1		-1	0	1	1	0	-1		-1	1	0	1	-1		-1	1	0	0	1	-1
N = 7	Num	-1	-2	-3	0	3	2	2		-1	2	-5	0	5	-2	2		-1	-2	1	0	-1	2	2		-1	-2	0	0	0	2	2		
	Dem	-1	-1	1	2	1	-1	-1		-1	1	1	-2	1	1	-1		-1	-1	1	2	1	-1	-1		-1	-1	1	2	1	-1	-1		
N = 8	Num	-1	0	1	0	0	-1	0	2		-1	0	0	0	0	0	2		-1	0	0	1	-1	0	0	2		-1	0	0	0	0	0	2
	Dem	-1	0	1	0	0	1	0	-1		-1	0	1	0	0	1	0	-1		-1	0	0	1	1	0	0	-1		-1	0	1	0	0	-1
N = 9	Num	-1	2	-1	0	0	0	1	-2	2		-1	2	-1	1	0	-1	1	-2	2		-1	-2	-2	0	0	0	2	2	2		-1	2	2
	Dem	-1	1	0	-1	2	-1	0	1	-1		-1	1	0	-1	2	-1	0	1	-1		-1	-1	0	1	2	1	0	-1	-1		-1	-1	
N = 10	Num	-1	0	1	0	0	0	0	-1	0	2		-1	0	-1	0	0	0	0	1	0	2		-1	0	0	0	0	0	0	0	0	2	
	Dem	-1	0	1	0	0	0	0	1	0	-1		-1	0	0	1	1	0	0	0	1	0	0		-1	0	1	0	0	0	0	1	0	-1
N = 11	Num	-1	0	-1	-2	0	0	0	2	1	0	2		-1	0	-1	-1	0	0	0	1	1	0	2		-1	-2	0	0	0	0	0	0	2
	Dem	-1	0	0	-1	1	2	1	-1	0	0	-1		-1	0	0	0	-1	1	2	1	-1	0	0	-1		-1	-1	1	0	0	2	0	0
N = 12	Num	-1	0	1	0	0	0	0	0	0	-1	0	2		-1	0	1	0	0	0	0	0	0	0	0	-1	0	2		-1	0	2		
	Dem	-1	0	1	0	0	0	0	0	0	1	0	-1		-1	0	1	0	0	0	0	0	0	0	0	1	0	-2		-1	0	0	-1	

The algorithms with letters (a) are better, more accurate, more robust and more stable for the random noise. The tests show that the optimum phase-shift interval with which the algorithm gives minimum errors for the noise is near $\pi/2$ radians (Fig. 1).

4. Change $\phi \in [0, \pi/2]$ to $\phi^* \in [-\pi, \pi]$

Because of the character of the evaluation algorithms, only phase values $\phi \in [0, \pi/2]$ were calculated. For unequivocal determination of the wrapped phase values ϕ it

was necessary to test four values, ϕ , $-\phi$, $\phi - \pi$, and $-\phi + \pi$, using values of I_k and small systems. With this, the value $\phi^* \in [-\pi, \pi]$ was obtained.³⁻⁶⁾ In case $N = 5$, with I_1, I_2, I_3, I_4 , and I_5 , δ was found in the first equation and the values, ϕ , $-\phi$, $\phi - \pi$, and $-\phi + \pi$ were attributed to ϕ^* in order to test the other equation; I_a was found using a second equation. As an example, for each (x, y) the four values, ϕ , $-\phi$, $\phi - \pi$ and $-\phi + \pi$ were tested in (addition and subtraction of first, last and middle frames, the I_k)

$$\text{even } N = 4 \begin{cases} \cos(\delta/2) = \pm \sqrt{\frac{I_1 - I_4}{4(I_2 - I_3)}} \\ I_1 - I_4 = 2I_a \sin(\phi^*) \sin(3\delta/2) \\ I_2 - I_3 = 2I_a \sin(\phi^*) \sin(\delta/2) \\ (I_1 + I_4) - (I_2 + I_3) = 2I_a \cos(\phi^*) [\cos(3\delta/2) - \cos(\delta/2)] \\ I_1 - I_3 = I_a [\cos(\phi^* - 3\delta/2) - \cos(\phi^* + \delta/2)] \end{cases}, \quad (17)$$

$$\text{odd } N = 5 \begin{cases} \cos(\delta) = \frac{I_1 - I_5}{2(I_2 - I_4)} \\ I_1 - I_5 = 2I_a \sin(\phi^*) \sin(2\delta) \\ I_2 - I_4 = 2I_a \sin(\phi^*) \sin(\delta) \\ I_1 + I_5 - 2I_3 = 2I_a \cos(\phi^*) [\cos(2\delta) - 1] \\ I_2 + I_4 - 2I_3 = 2I_a \cos(\phi^*) [\cos(\delta) - 1] \end{cases}. \quad (18)$$

In a different approach, for unambiguous determination of the wrapped phase values, it is necessary to test four values, ϕ , $-\phi$, $\phi - \pi$, and $-\phi + \pi$, using values of I_k and to solve small nonlinear systems (Newton–Raphson method). For each angle, ϕ , $-\phi$, $\phi - \pi$, and $-\phi + \pi$, the nonlinear system by Newton–Raphson in eq. (19) can be solved, getting the values of I_m , I_a , and δ .

$$\begin{cases} I_1 - \left(I_m + I_a \cos \left[\phi^* + \left(\frac{2.1 - N - 1}{2} \right) \delta \right] \right) = 0 \\ I_2 - \left(I_m + I_a \cos \left[\phi^* + \left(\frac{2.2 - N - 1}{2} \right) \delta \right] \right) = 0 \\ I_3 - \left(I_m + I_a \cos \left[\phi^* + \left(\frac{2.3 - N - 1}{2} \right) \delta \right] \right) = 0 \end{cases} \quad (19)$$

Test the values of I_m , I_a and δ , in eq. (20) and find the correct angle $\phi^* \in [-\pi, \pi]$.

$$\begin{cases} I_4 - \left(I_m + I_a \cos \left[\phi^* + \left(\frac{2.4 - N - 1}{2} \right) \delta \right] \right) = 0 \\ \dots \\ I_N - \left(I_m + I_a \cos \left[\phi^* + \left(\frac{2N - N - 1}{2} \right) \delta \right] \right) = 0 \end{cases} \quad (20)$$

5. Testing and Analysis of Error

The phase ϕ^* obtained from the Phase Shifting Algorithm above is a wrapped phase, which varies from $-\pi/2$ to $\pi/2$. The relationship between the wrapped phase and the unwrapped phase may thus be stated as

$$\Psi(x, y) = \phi^*(x, y) + 2\pi j(x, y), \quad (21)$$

where j is an integer number, ϕ^* is a wrapped phase and ψ is an unwrapped phase.

The next step is to unwrap the wrapped phase map.¹³⁾ When unwrapping, several of the phase values should be shifted by an integer multiple of 2π . Unwrapping is thus adding or subtracting 2π offsets at each discontinuity encountered in phase data. The unwrapping procedure consists of finding the correct field number for each phase measurement.⁹⁻¹¹⁾

The modulation phase ψ obtained by unwrapping physically represents the fractional fringe order numbers in the Moire images. The shape can be determined by applying the out-of-plane deformation equation for Shadow Moire:

$$Z(x, y) = \frac{p(\Psi(x, y)/2\pi)}{\tan \alpha + \tan \beta}, \quad (22)$$

where $Z(x, y)$ = elevation difference between two points located at the body surface to be analyzed; p = frame period; α = light angle; β = observation angle.

The experiments were carried out using a square wave grating with 1 mm frame grid period, light source was common white of 300 W with no use of plane waves, light angle (α) and observation angle (β) were 45° , the object surface was white and smooth and the resolution of photo was one mega pixel. The phase stepping was made by displacing the grid in the horizontal direction in fractions of millimetres (Fig. 2).

To test the new equations for phase calculation, they were used with the technique of Shadow Moire¹²⁾ for an object with known dimensions and the average error evaluated by

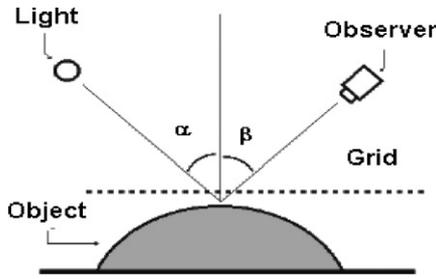


Fig. 2. Layout of experiment.

eq. (23). This process was started with 4 images, again with 5, then 6 and so on. The idea was to show that with the increasing number of images the average error tends to decrease. Figure 3 shows this procedure.

$$\text{Error median } (E) = \frac{1}{M} \sum_{i=1}^M |Z_i^c - Z_i|, \quad (23)$$

where M is number of pixels of the image, Z_i^c is the exact value of the size of the object being measured and Z_i is value measured by the new equation.

To compare the new equations for calculating the phase, 21 sets of 16 photos each were selected. Each set was computed using the average error of 4 to 16 images and using

equations to evaluate what the number of images was. The average errors were estimated and was 21 sets using 4 to 16 images in each set ($\mu_4, \mu_5, \mu_6, \dots, \mu_{16}$). The hypothesis of testing the difference in the means $\mu_A - \mu_B$ of two normal populations is being considered at the moment. A more powerful experimental procedure is to collect the data in pairs—that is, to make two hardness readings on each specimen, one with each tip. The test procedure would then consist of analyzing the differences between hardness readings on each specimen. If there is no difference between tips, the mean of the differences should be zero. This test procedure is called the paired t -test.¹³⁾ Specifically, testing $H_0: \mu_A - \mu_B = 0$ against $H_1: \mu_A - \mu_B \neq 0$. Test statistics is $t_0 = D/(S_D/\sqrt{21})$ where D is the sample average of the differences and S_D is the sample standard deviation of these differences. The rejection region is $t_0 > t_{\alpha/2,20}$ or $t_0 < -t_{\alpha/2,20}$.

After doing the statistical test ($H_0: \mu_A - \mu_B = 0$ against $H_1: \mu_A - \mu_B \neq 0$) it was noticed that one cannot reject the zero hypothesis when using different equations, which have the same number of images. Also, the null hypothesis can be rejected when using different equations with different numbers of images with level of significance ($\alpha = 0.05$). It was concluded that the equations for phase calculation with a greater number of images are more accurate than those that have a smaller number of images (Table. 4).

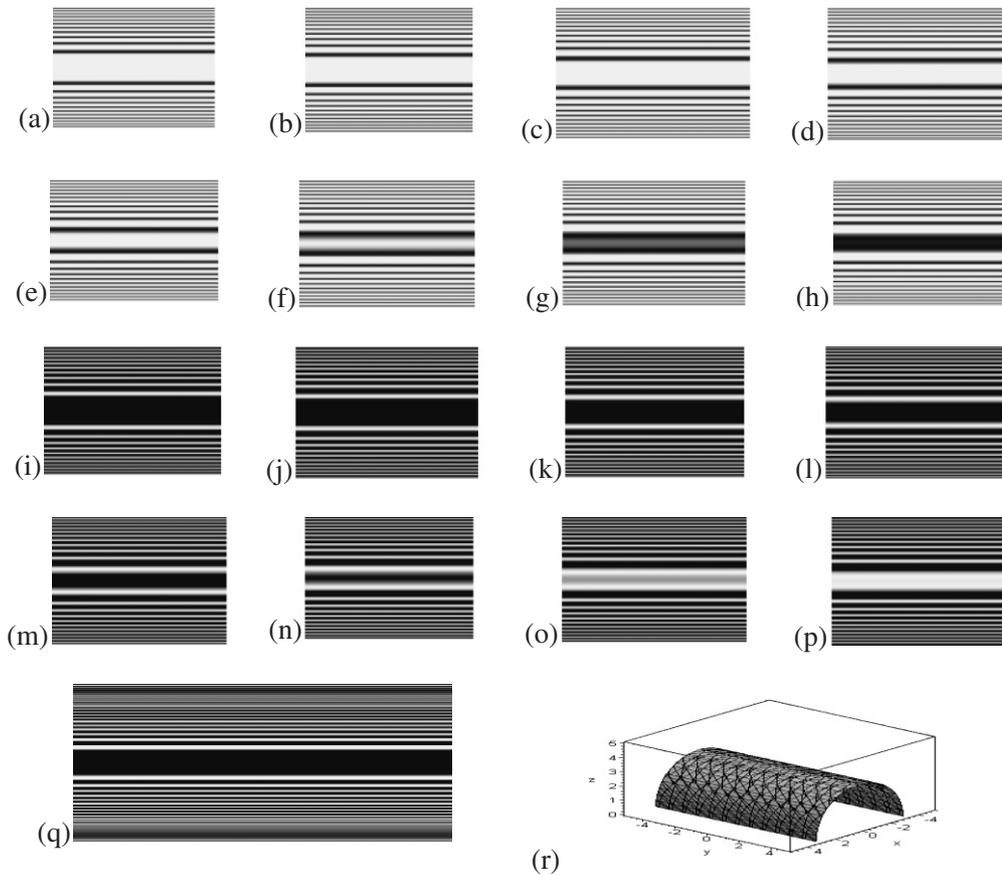


Fig. 3. One set of photos of 1 megabyte. Original shadow moire images. 16-frame phase-shifting algorithm. (a-p). Wrapped phase (q). Result in 3D (r). Semi-cylinder of a motor with diameter of 6 cm, length of 12 cm and frame period of grid with 1 mm.

Table 4. Testing hypotheses about the difference between two means with paired *t*-test, $H_0: \mu_A - \mu_B = 0$ against $H_1: \mu_A - \mu_B \neq 0$. The *P*-value is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

P-Value	Number of Images and Formula																															
	4	5	6				7				8			9			10			11		12		13		14		15				
Number of Image and Formula	a	a	a	b	c	d	e	a	b	c	d	a	b	c	d	a	b	c	a	b	c	a	b	c	a	b	a	b	a			
4	a																															
5	a	0%																														
6	a	0%	0%																													
	b	0%	0%	65%																												
	c	0%	0%	80%	77%																											
	d	0%	0%	39%	13%	19%																										
7	a	0%	0%	0%	0%	0%	0%																									
	b	0%	0%	0%	0%	0%	0%	67%																								
	c	0%	0%	0%	0%	0%	0%	68%	96%																							
	d	0%	0%	0%	0%	0%	0%	26%	67%	59%																						
8	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%																					
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	69%																				
	c	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	65%	98%																			
	d	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	75%	46%	48%																		
9	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%																		
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	26%																	
	c	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	79%	22%																
10	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%															
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	71%														
	c	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	65%	84%													
11	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%												
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	32%											
	c	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	18%	84%									
12	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%									
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	73%								
13	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%							
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	30%						
14	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%						
	b	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	78%					
15	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%		
16	a	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%		

6. Conclusion

This paper deals with the algorithms for phase calculation in measurement with image methods using the phase shifting technique. It describes several multistep phase shifting algorithms with the constant but unknown phase step between the captured intensity frames. The new equations are shown to be capable of processing the optical signal of Moire images. These techniques are very precise, easy to use, and have low cost. The results show that new equations were precise and accurate. On the basis of the performed error analysis it can be concluded that the new equations are very good phase calculation algorithms. These equations also seem to be very accurate and stable phase shifting algorithms with the unknown phase step for a wide range of phase step values. The metric analysis of the considered system demonstrated that its uncertainties of measurement depend on the frame period of the grid, of the resolution in pixel of photos and on the number of frames. However, the uncertainties of measurement of the geometric parameters and the phase still require attention.

In theory, if we have many frames, the measurement errors become very small. The measurement results obtained by the optical system demonstrate its industrial and engineering applications.

Acknowledgements

The authors appreciate the generous support of the Pontificia Universidade Catolica de Minas Gerais—PUCMINAS, as well as of the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico—CNPq—“National Counsel of Technological and Scientific Development”.

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