COBEM 87 — IX Congresso Brasileiro de Engenharia Mecânica, Florianópolis, SC (Dezembro 1987)
9th Brazilian Congress of Mechanical Engineering

Local Unilateral Contact Problems in Beams Using a Higher Order Theory

CARLOS ALBERTO DE CAMPOS SELKE*
Depto. de Engenharia Mecânica — UFSC
Cx. Postal 476 - 88049 - Florianópolis - SC

NOBORU KIKUCHI
Dept. Mechanical Engineering and Applied Mechanics - The University of Michigan
Ann Arbor, Michigan 48109 - 2125 - USA

SUMMARY

An incremental finite element formulation uses the updated lagrangian approach for the problems of beams undergoing large deflections with unilateral constraints. This formulation which approximates the geometrical nonlinearities as well as the nonlinearities arising from the contact constraints, is particularized for the Essenburg beam theory where the effects of transverse shear deformation and transverse normal strain are considered, and for the classic beam theory. Two typical problems of bending and buckling of beams constrained by lateral walls are solved as numerical examples.

INTRODUCTION

The problem of bending of beams with lateral restrictions was first considered by Timoshenko [1], using the Euler-Bernoulli hypothesis, with the beam being partially constrained by a rigid cylindrical surface or by a rigid horizontal foundation. Similar studies for the bending of circular plates, constrained by horizontal foundations parallel to the underformed middle surface of the plate, were performed by Girkmann [2] and Timoshenko and Woinowsky-Krieger [3], using the classic plate theory, motivated by problems in the bottom plate of liquid containers.

The problem of buckling of straight beams with lateral restrictions was first studied by Link [4] using the classic beam theory assumptions. In his work, Link examines the problem of a straight elastic simply supported beam axially compressed and placed between two rigid walls which are separated by a small distance.

The solution of contact problems in beams using the classical theory had some limitations such as the incorrect prediction of the tractions at the periphery of the contact region and the failure to predict the regions of separation after correctly predicting the increase in the contact region for a monotonically increasing load. Trying to remove these difficulties, Essenburg [5] introduced a higher order beam theory that takes into account the effect of the transverse normal strain as well as the effect of the transverse shear deformation, and the regions of separation were satisfactory determired.

The numerical solution of unilateral contact problems in continuum mechanics has been developed using the penalty method and the regularity method to solve the variational inequality that characterizes the contact problem and Kikuchi and Oden [6] have presented an extensive description of the contact problems in elastostatics. Numerical analyses of beam bending contact problems based on the classic beam theory assumptions were performed by Kikuchi [7] and Ohtake et al. [8]. Stein and Wriggers [9] have presented a finite element formulation for the stability of rods with unilateral constraints, considering the transverse shear deformation effect.

This work presents an incremental finite element formulation for the problems of beams with local unilateral constraints, undergoing large deflections. The Essenburg beam theory is considered and the penalty method is used to solve the variational inequality. A comparison with the results obtained from the same formulation considering the classical beam theory is provided.

* Member of ABCM

THE INCREMENTAL PRINCIPLE OF VIRTUAL WORK

The global representation of the problem of deformation of an elastic body Ω, caused by body forces and by forces applied at the boundary Γ_F, constrained by a rigid surface Γ_c, is obtained by writing the Principle of Virtual Work for a general reference configuration Ω_R, with t^F_{Ri} and t^c_{Ri} being the components of the surface traction vector at Γ_F and Γ_c, referred to Γ_{FR} and Γ_{cR}, respectively, ρ_R being the density of the body at the reference configuration, S_{ij} being the components of the 2nd Piola-Kirchhoff stress tensor, defined at the reference configuration, and E_{ij} being the components of the Green strain tensor.

Let the body Ω, at the current configuration Ω_t, suffer an incremental displacement Δu, caused by increments in the body forces Δf and in the applied surface tractions Δt^F_R and Δt^c_R, producing then an increment ΔS of the stress tensor. Neglecting the higher order terms, the incremental form of the principle of virtual work can be written similarly as given by Washizu [10].

THE CONTACT CONDITIONS

Consider the body Ω as a beam. The problem of a beam placed between two rigid surfaces and subjected to a deformation process, as shown by Figure 1, characterizes the Signorini's problem (Duvaut and Lions [11]).

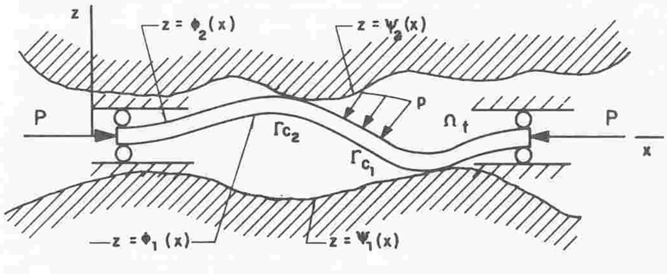

Figure 1 : Beam undergoing a constrained deformation process.

A point on the top or bottom surface of the beam has the coordinates (x, y(x)) and under the deformation process moves to a new position $(x + u_x(x, y(x)), y(x) + u_z(x, y(x))$ where u_x and u_z are the components of the displacement of the point on the surface. The beam has its upper and lower surfaces not allowed to move further than the rigid surfaces, i.e.,

$$\phi_1(x) + u_z(x, \phi_1(x)) \geq \psi_1 \; (x + u_x(x, \phi_1(x))) \quad (1)$$

$$\phi_2(x) + u_z(x, \phi_2(x)) \leq \psi_2 \; (x + u_x(x, \phi_2 \; (x))) \quad (2)$$

where $(x, \phi_1(x)) \in \Gamma_{c1}$ and $(x, \phi_2(x)) \in \Gamma_{c2}$. Equations (1) and (2) are the no-penetration conditions.

Consider the beam in the current configuration being displaced by small increments Δu_i. These incremental displacements are introduced in (1) and (2) and after a linearization process with respect to Δu_i is performed, based on a Taylor series expansion, the incremental contact conditions are written in the form of the Signorini's boundary condition as [11]

$$\Delta u_{N1} - \Delta g_{N1} \leq 0 \qquad \text{on } \Gamma_{c1} \quad (3)$$

$$\Delta u_{N2} - \Delta g_{N2} \leq 0 \qquad \text{on } \Gamma_{c2} \quad (4)$$

where $\Delta u_{N\alpha} = \underset{\sim}{N}_\alpha \cdot \Delta \underset{\sim}{u}$ with $\underset{\sim}{N}_\alpha$ being the inward normal to the rigid surface $z = \psi_\alpha(x)$, having the components

$$N_{\alpha x} \; (y) = \pm \frac{\partial\psi\alpha/\partial y}{[1 + (\partial\psi\alpha/\partial y)^2]^{1/2}} \quad (5)$$

$$N_{\alpha z} \; (y) = \mp \frac{1}{[1 + (\partial\psi\alpha/\partial y)^2]^{1/2}} \quad (6)$$

where $y = x + u_x(x, \mp h/2)$ and $\Delta g_{N\alpha}$ are the incremental gap functions given by

$$\Delta g_{N1} = \frac{-h/2 - u_z(x,-h/2) - \psi_1(y)}{[1 + (\partial\psi_1/\partial y)^2]^{1/2}} \quad (7)$$

$$\Delta g_{N2} = \frac{\psi_2(y) - h/2 - u_z \; (x, \; h/2)}{[1 + (\partial\psi_2/\partial y)^2]^{1 \; 2}} \quad (8)$$

Then, the stress contact conditions at the boundaries Γ_{c1} and Γ_{c2} can be established as

$$t_{N1} \; (\underset{\sim}{y} + \Delta \underset{\sim}{u}) = 0 \quad \text{if} \quad \Delta u_{N1} - \Delta g_{N1} < 0$$
$$t_{N1} \; (\underset{\sim}{y} + \Delta \underset{\sim}{u}) \quad 0 \quad \text{if} \quad \Delta u_{N1} - \Delta g_{N1} = 0 \quad (9)$$

and

$$t_{N2} (\underset{\sim}{y} + \Delta \underset{\sim}{u}) = 0 \quad \text{if} \quad \Delta u_{N2} - \Delta g_{N2} < 0$$
$$t_{N2} (\underset{\sim}{y} + \Delta \underset{\sim}{u}) = 0 \quad \text{if} \quad \Delta u_{N2} - \Delta g_{N2} = 0 \quad (10)$$

with $\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{u}$ and

where $t_{N\alpha}(\underset{\sim}{y} + \Delta \underset{\sim}{u})$ are the normal components of the Cauchy stress vector and its tangential components are $t_{T\alpha} = 0$, since the rigid surfaces are frictionless.

THE VARIATIONAL INEQUALITY AND THE PENALTY FORMULATION

In the incremental principle of virtual work the contact terms are represented by quantities referred to the reference configuration, whereas the contact conditions are given in the current configuration. The proper consideration of the contact terms in the incremental lagrangian formulation is done in a procedure similar to the one presented by Kikuchi and Oden [12], yielding to a variational inequality [15].

This inequality can be solved by using the exterior penalty method which includes the contact constraints in an equality that approximates the mentioned inequality. This idea was first introduced by Courant [13] and is equivalent to replace the rigid surfaces that produce the contact conditions by sets of very stiff springs, i.e., the incremental contact pressure can be written as

$$\Delta t_N^c = - \frac{1}{\mu} \; (\Delta u_N - \Delta g_N) \text{ on } \Gamma_c \quad (11)$$

where μ is the penalty factor. The existence of the solution and the convergence to the original problem as $\mu \to \infty$ are given by Kikuchi and Song [14]. The resulting equality is

$$\int_{\Omega_R} (\Delta S_{ji} \delta E_{ij} + S_{ji} \Delta u_{k,i} \delta u_{k,j}) \, d\Omega_R + \frac{1}{\mu} \int_{\Gamma_{cR}} J F_{ij}^{-1} n_{Rj} n_k \Delta u_k \delta u_i d\Gamma_R + \int_{\Gamma_{cR}} J F_{ik}^{-1} n_{Rj} t_N^c \Delta u_{k,j} \delta u_i d\Gamma_R - \int_{\Gamma_{cR}} J F_{ij}^{-1} n_{Rj} t_N^c \Delta u_{k,k} \delta u_i d\Gamma_R = \int_{\Omega_R} \Delta b_{Ri} \delta u_i d\Omega_R + \int_{\Gamma_{FR}} \Delta t_{Ri}^F \delta u_i d\Gamma_R + \frac{1}{\mu} \int_{\Gamma_{cR}} J F_{ij}^{-1} n_{Rj} \Delta g_N \delta u_i d\Gamma_R + R \quad (12)$$

where the relations $\Delta J = J \Delta u_{i,i}$, $\Delta F_{ij}^{-1} = - F_{ik}^{-1} \Delta u_{k,j}$ and $\Delta u_N = n_i \Delta u_i$ were used, J being the determinant of the deformation gradient F_{ij}, n_{Rj} the components of the outward normal to the contact surface Γ_{cR}, and R is the residue.

THE UPDATED LAGRANGIAN FORMULATION

Consider the case when the current configuration Ω_t is taken as the reference configuration Ω_R. The relations $J = 1$, $F_{ij}^{-1} = \delta_{ij}$ and $S_{ij} = \sigma_{ij}$ apply, σ_{ij} being the components of the Cauchy stress tensor. The incremental principle of virtual work is written as

$$\int_{\Omega} (\Delta S_{ji} \delta \varepsilon_{ij} + \sigma_{ji} \Delta u_{k,i} \delta u_{k,i}) \, d\Omega + \frac{1}{\mu} \int_{\Gamma_c} n_i n_k \Delta u_k \delta u_i d\Gamma = \int_{\Omega} \Delta b_i \delta u_i d\Omega + \int_{\Gamma_F} \Delta t_i^F \delta u_i d\Gamma + \frac{1}{\mu} \int_{\Gamma_c} n_i \Delta g_N \delta u_i d\Gamma + R \quad (13)$$

where ε_{ij} are the components of the infinitesimal strain tensor. With the consideration of the incremental constitutive law $\Delta S_{ij} = C_{ijk\ell} \, \Delta\varepsilon_{k\ell}$, where $C_{ijk\ell}$ are the components of the elasticity tensor, the discretization procedure of (13) is done by considering the beam being divide into E elements, and inside of each element the incremental and virtual displacements are interpolated in terms of the incremental and virtual nodal displacements. Taking into consideration the arbitrariness of $\delta \underset{\sim}{u}$, one can write for each element

$$[K_{\alpha I \beta J}] \; \{\Delta u_{\beta J}\} = \{\Delta f_{\alpha I}\} \quad (14)$$

where $[K_{\alpha I \beta J}] = [K_{\alpha I \beta J}^{\Delta u} + K_{\alpha I \beta J}^{\sigma}]$, being $[K_{\alpha I \beta J}^{\Delta u}]$ the incremental stiffness matrix, similar to the stiffness matrix obtained in the linear analysis, and $[K_{\alpha I \beta J}^{\sigma}]$ is the initial stress stiffness matrix.

APPLICATIONS: ESSENBURG AND CLASSIC BEAM THEORIES

The formulation presented can be used for any type of beam theory. The use of higher order theories would produce a greater number of degrees of freedom in the finite element equations. Consider the Essenburg beam theory, which includes the effects of transverse shear deformation and transverse normal strain with its displacement field being

$$u_x(x,z) \doteq u(x) + z \; \Theta_x(x)$$
$$u_z(x,z) \doteq w(x) + z \; \Theta_z(x) + z^2 \xi_z(x) \quad (15)$$

where u_x is the axial displacement and u_z is the displacement of any point of the beam, u is the axial displacement and w is the vertical displacement of a point on the neutral axis of the beam, Θ_x is the measure of the rotation of the cross section, and Θ_z and ξ_z are kinematic measures related to the deformation along the thickness.

The discretization procedure of the expression of the principle of virtual work for the Essenburg beam theory is done by interpolating the incremental and virtual displacements for two-node elements of lenght L by

$$\Delta u = N^1_\alpha(x)\Delta u_\alpha \qquad \delta u = N^1_\alpha(x)\delta u_\alpha$$
$$\Delta w = N^2_\alpha(x)\Delta w_\alpha \qquad \delta w = N^2_\alpha(x)\delta w_\alpha$$
$$\Delta\Theta_x = N^3_\alpha(x)\Delta\Theta_{x\alpha} \qquad \delta\Theta_x = N^3_\alpha(x)\delta\Theta_{x\alpha}$$
$$\Delta\Theta_z = N^4_\alpha(x)\Delta\Theta_{z\alpha} \qquad \delta\Theta_z = N^4_\alpha(x)\delta\Theta_{z\alpha}$$
$$\Delta\xi_z = N^5_\alpha(x)\Delta\xi_{z\alpha} \qquad \delta\xi_z = N^5_\alpha(x)\delta\xi_{z\alpha} \qquad (16)$$

where $N^n_\alpha(x)$, n = 1,5 are the linear shape functions given by

$$N^n_1(x) = 1 - \frac{x}{L} \qquad N^n_2(x) = \frac{x}{L} \qquad (17)$$

leading to a finite element equation for each element similar to (14).

Consider now the classic beam theory, where the Euler - Bernoulli hypothesis is adopted, i.e., the normals perpendicular to the centroidal axis before the deformation remain straight and perpendicular to the neutral axis after the deformation and do not suffer changes in their lenght. The displacement field is given by

$$u_x(x,z) \doteq u(x) - z\,\frac{dw}{dx}\,(x)$$
$$u_z(x,z) \doteq w\,(x) \qquad (18)$$

where again u_x is the axial displacement and u_z is the transversal displacement of any point of the beam, u is the axial displacement and w is the vertical displacement of a point on the neutral axis, while dw/dx is the measure of the rotation of the cross-section.

The well-known discretization procedure for the classic beam theory is done by interpolating the incremental and the virtual displacements for two-node elements of lenght L by

$$\Delta u = N^1_\alpha(x)\Delta u_\alpha \qquad \delta u = N^1_\alpha(x)\delta u_\alpha$$
$$\Delta w = N^2_\alpha(x)\Delta w_\alpha + N^3_\alpha(x)\Delta\Theta_\alpha \qquad \delta w = N^2_\alpha(x)\delta w_\alpha + N^3_\alpha(x)\delta\Theta_\alpha \qquad (19)$$

where $N^n_\alpha(x)$ represents the well-known shape functions

$$N^1_1(x) = 1 - x/L \qquad N^1_2(x) = x/L$$
$$N^2_1(x) = 1 - 3x^2/L^2 + 2x^3/L^3 \qquad N^2_2(x) = 3x^2/L^2 - 2x^3/L^3$$
$$N^3_1(x) = x - 2\,x/L + x/L \qquad N^3_2(x) = -x^2/L + x^3/L^2 \qquad (20)$$

and again a similar expression to (14) is obtained for each element.

The detailed finite element equations, similar to (14), for the two particular cases described above can be found in [15].

EXAMPLES AND DISCUSSION OF RESULTS

Two typical problems of bending and buckling of beams with unilateral constraints were solved using the above formulations.

A simply supported beam subjected to concentrated moments M_0, as shown in Figure 2, is bent against a rigid foundation located at the position $z = \psi(x) = -d$.

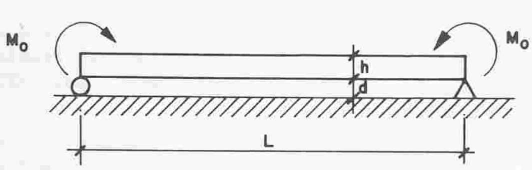

Figure 2 : Beam bending against a rigid foundation.

This problem was solved numerically, with the following characteristics: $E = 2.1 \times 10^4$ kN/cm^2, $\nu = 0.29$, L = 2 cm, b = 0,2 cm, h = 0.2 cm and d = 0.08 cm. The values of the displacement of the middle point of the beam as well as the corresponding values of M_0 are presented in Table 1. The notations (C*) and (E*) highlight the values of M_0 when the initial contact between the lower surface of the beam and the rigid foundation ocurred, for the classic and the Essenburg beam theories, respectively. It should be noted that, according to analytical solutions provided by Essenburg [5], the expected values of M_0 when the initial contact occurs are, for the classic beam theory M_0 = 0.4480 kN cm and for the Essenburg beam theory M_0 = 0.4493 kN. cm. The notation (E**) highlight the value of M_0 for the beginning of the separation between the lower surface of the beam and the rigid foundation, inside the contact region. Theoretically, the classic beam theory fails to predict this region of separation and this fact was confirmed by the numerical simulation. The analytic solution that the Essenburg beam theory [5] gives for the value of M_0 to start developing the region of separation is M_0 = 3.4874 kN.cm. The deformation behavior of the beam is illustrated in the sequences shown in Figures 3 and 4. The sequences were composed by samples taken form the several incremental steps applied.

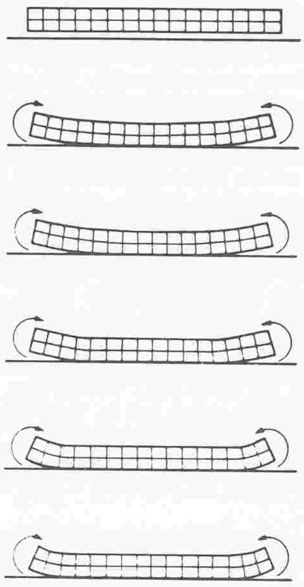

Figure 3 : Deformation process for a beam bending against rigid foundation (Classical beam theory)

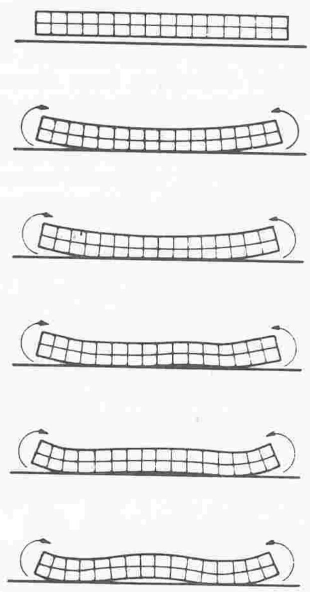

Figure 4 : Deformation process for a beam bending against a rigid foundation. (Essenburg beam theory)

The appearance of the region of separation is well described by using the formulation based on the Essenburg beam theory, while the formulation based on the classic beam theory does not show it, as expected. Comparing the values of M_0 obtained numerically with the analytical values predicted em [5], one can see a good agreement of the results obtained with the finite element formulations and the analytical results.

Displacement of the middle point of the beam		
M_0 (kN. cm)	Classic Theory (cm)	Essenburg Theory (cm)
0.4000	-0.0715	-0.0706
0.4475	-0.0799	-0.0790
0.4485(C*)	-0.0800	-0.0792
0.4495(E*)	-0.0800	-0.0800
0.5000	-0.0800	-0.0800
2.5000	-0.0800	-0.0800
3.0000	-0.0800	-0.0800
3.5000(E**)	-0.0800	-0.0799
5.0000	-0.0800	-0.0771
5.1800	-0.0800	-0.0769
7.5000	-0.0800	-0.0735
10.0000	-0.0800	-0.0703

Table 1: Displacement of the middle point of the beam (Concentrated moment M_0)

Displacement of the middle point of the beam		
P_0 (kN)	Classic Theory (cm)	Essenburg Theory (cm)
0.0000	0.0302	0.0302
0.4150	0.0444	0.0443
0.4270	0.1479	0.1472
0.4275(C*)	0.1500	0.1489
0.4280(E*)	0.1500	0.1500
0.5000	0.1500	0.1500
1.7500	0.1500	0.1500
5.0000	0.1500	0.1500
7.0000(C**)	0.1490	0.1500
7.1000	0.1374	0.1500
7.1500(E**)	0.1256	0.1494
7.5000	0.0420	0.0680
10.0000	-0.1500	-0.1500

Table 2: Displacement of the middle point of the beam (Axial load P_0)

A simply supported beam subjected to axial loads P_0, as shown in Figure 5, buckles at its critical load; this phenomenon is simulated by the introduction of a small imperfection in a way that the beam is deforming in the transversal direction since the beginning of the application of the axial load. The usual deformation mode of the beam for such loading conditions is not achieved because of the presence of the two walls. At $z = \psi_1(x) = -d/2$ and $z = \psi_2(x) = d/2$.

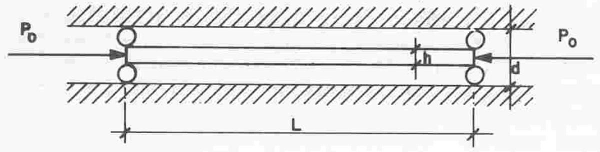

Figure 5 : Beam axially loaded constrained by lateral walls.

This problem was solved numerically with the following characteristics: $E = 2.1 \times 10^4$ kN/cm^2, $\nu = 0.29$, $L = 2$ cm, $b = 0.1$ cm, $h = 0.1$ cm and $d = 0.3$ cm. The values of the displacement of the middle point of the beam as well as the corresponding values of the axial load P are presented in Table 2. The notations (C*) and E*) highlight the values of the axial load P_0 when the initial contact between the upper surface of the beam and the rigid lateral wall ocurred, for the classic and the Essenburg beam theories, respectively. The value of the Euler critical load for the first mode of the unconstrained beam, using the classic beam theory, is $P_0 = 0.4318$ kN. In the analytical solutions proposed by Link [4], the expected value of P_0 when the initial contact occurs, is the Euler critical load. The continously increasing axial load P_0 produces the increase in the lenght of the part of the beam leaning against the rigid wall until this part of the beam snaps back. The notations (C**) and (E**) highlight the values, for the classic and the Essenburg beam theories, respectively, of P_0 for the snap-through of the part of the beam laying against the lateral rigid wall. In the analytical solutions proposed by Link [4], using the classic beam theory, the expected value of P_0 for the snap-through of the middle part of the beam is $P_0 = 6.9087$ kN. The deformation behavior of the beam is illustrated in the sequences shown in Figures 6 and 7.

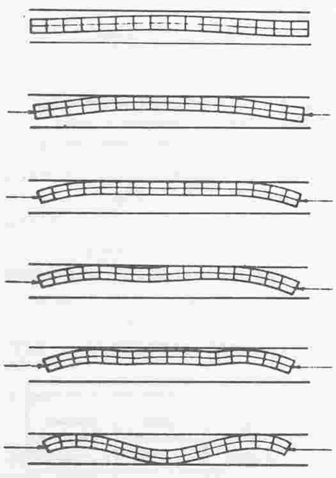

Figure 6: Deformation process for the buckling of a beam constrained by lateral walls (Classical theory)

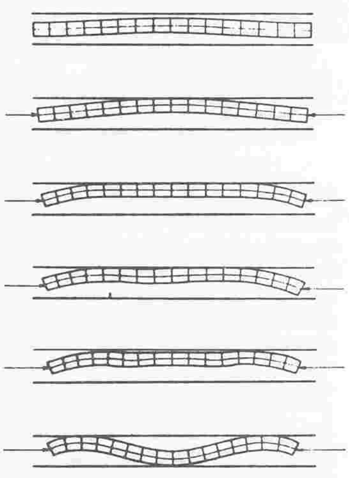

Figure 7: Deformation process for the buckling of a beam constrained by lateral walls. (Essenburg theory)

Again, the sequences were composed by samples taken among the several incremental steps applied. The ocourence of the secondary bifurcation, due to a limit point or snap-through instability, predicted by Link [4] was detected and confirms the capability of the presented numerical formulations in represent such phenomenon. After this load is reached, the beam reaches the other wall, with the load kept constant. The similarity between the solutions obtained using the two theories, leads to the question of the necessity of considering of not the higher order terms for this class of problems. All the results indicate that the consideration of the higher order terms is not a fundamentally important factor in determining the deformation behavior of the beam axially loaded, constrained in a channel.

CONCLUSIONS

An incremental lagrangian formulation for local unilateral contact problems of beams was presented leading to incremental finite element formulations for the classic and the Essenburg beam theories. The results obtained have shown that the deformation behavior of beams in bending or buckling with unilateral constraints can be represented by the incremental finite element formulations presented.

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