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A COMPARISON OF THE MODIFIED LOCAL GREEN'S FUNCTION METHOD (MLGFM) AND THE FINITE ELEMENT METHOD (FEM) FOR SOME PROBLEMS IN 3-D ELASTICITY

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ABSTRACT: In this paper, the MLGFM is compared to the FEM for the solution of selected problems of 3-D elasticity. For this purpose, a FEM code was specially developed, in which special care was taken in order to use, as much as possible, the same routines employed in the MLGFM code. This, together with the fact that the same finite elements and meshes were always chosen when comparing the methods, guarantees reliable conclusions. The comparisons are done for displacements and stresses. In both, MLGFM and FEM, the evaluation is performed at the nodal locations, since the MLGFM provides stress results directly at these points, without using extrapolation techniques. The analysis employs hexaedric elements of 20 and 27 nodes in the domain and quadratic elements of 8 and 9 nodes on the boundary.

1. INTRODUCTION

The MLGFM is a numerical method which is closely related to the Galerkin formulation of the Boundary Element Method (BEM). The main difference between these two techniques is that the former does not require an analytical form of a fundamental solution.In this sense, the applicability of the MLGFM is broader than the one of the traditional BEM.

The procedure that the MLGFM employs to overcome the need of a fundamental solution is to replace it by locally determined Green's functions projections which are evaluated with the help of the FEM. The boundary conditions adopted for this purpose are chosen in such a way that the integral equations that describe the problem get simplified.

The application of the MLGFM to elasticity problems was originally presented by Barbieri (1992) and Barbieri et al. (1992), where 2-D problems were analysed. Excellent results were achieved encouraging further research and application to the 3-D case. The first analysis involving 3-D problems was presented by Meira Jr. (1994).

In this paper the application of the MLGFM to 3-D elasticity problems is detailed and some numerical examples are displayed. The results are compared to the equivalent FEM ones.

2. DERIVATION OF THE MLGFM INTEGRAL EQUATIONS

This section describes the use of the MLGFM to solve the 3-D elasticity problem defined by the expression

$$\mathcal{A}\mathbf{u} = \mathbf{b} \quad \text{in } \Omega \tag{2.1}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial\Omega_1 \tag{2.2}$$

$$\mathcal{N}\mathbf{u} = \bar{\mathbf{T}} \text{ on } \partial\Omega_2 \tag{2.3}$$

where $\mathbf{u} = \{u_1 \quad u_2 \quad u_3\}^t$ is the generalized displacement vector; $\mathcal{A}, \mathcal{N}, \bar{\mathbf{u}}$ and $\bar{\mathbf{T}}$ are respectively, the differencial operator of 3-D elasticity, the associated Neumann operator, and the prescribed values for the generalized displacement vector and generalized forces. The vector $\mathbf{b} = \{b_1 \quad b_2 \quad b_3\}^t$ corresponds to external body loading.

Consider the adjoint operator $\mathcal{A}^*$ and the associated problem

$$\mathcal{A}^*\mathbf{G}(Q,P) = \delta(Q,P)\mathbf{I} \tag{2.4}$$

where $\delta(Q,P)$ is the Dirac delta generalized function, $\mathbf{I}$ is the identity tensor and $\mathbf{G}(Q,P)$ is a fundamental solution tensor which represents the displacement at a field point Q due to a unit source applied at point P.

Multiplying equation (2.1) by $\mathbf{G}(Q,P)^t$ and equation (2.4) by $\mathbf{u}(Q)^t$ results in

$$\mathbf{G}(Q,P)^t\mathcal{A}\mathbf{u}(Q) = \mathbf{G}(Q,P)^t\mathbf{b}(Q) \tag{2.5}$$

and

$$\mathbf{u}(Q)^t\mathcal{A}^*\mathbf{G}(Q,P) = \mathbf{u}(Q)^t\delta(Q,P) \tag{2.6}$$

Subtracting (2.5) from the transpose of (2.6) results in

$$\mathbf{u}(Q)\delta(Q,P) = [\mathcal{A}^*\mathbf{G}(Q,P)]^t\mathbf{u}(Q) - \mathbf{G}(Q,P)^t\mathcal{A}\mathbf{u}(Q) + \mathbf{G}(Q,P)^t\mathbf{b}(Q) \tag{2.7}$$

Integrating this equation on the domain Ω, with point P considered as fixed, one obtains

$$\mathbf{u}(P) = \int_\Omega [\mathcal{A}^*\mathbf{G}(Q,P)]^t\mathbf{u}(Q)d\Omega_Q - \int_\Omega \mathbf{G}(Q,P)^t\mathcal{A}\mathbf{u}(Q)d\Omega_Q + \int_\Omega \mathbf{G}(Q,P)^t\mathbf{b}(Q)d\Omega_Q \tag{2.8}$$

Next, applying the Gauss theorem to the first two integrals in (2.8) results in

$$\mathbf{u}(P) = -\int_{\partial\Omega} [\mathcal{N}^*\mathbf{G}(q,P)]^t\mathbf{u}(q)d\partial\Omega_q + \int_{\partial\Omega} \mathbf{G}(q,P)^t\mathcal{N}\mathbf{u}(q)d\partial\Omega_q + \int_\Omega \mathbf{G}(Q,P)^t\mathbf{b}(Q)d\Omega_Q \tag{2.9}$$

where $\mathcal{N}^*$ is the Neumann operator associated to the adjoint operator $\mathcal{A}^*$.

At this point it is convenient to define an operator named $\mathcal{N}'$ such that

$$G(q,P)^t N' u(q) = N' G(q,P)^t u(q) \tag{2.10}$$

The quantity defined by (2.10) can be added and subtracted from (2.9) resulting in

$$u(P) = -\int_{\partial\Omega}[(N^* + N')G(q,P)]^t u(q)\, d\partial\Omega_q + \int_{\partial\Omega} G(q,P)^t (N + N')u(q)\, d\partial\Omega_q + \int_{\Omega} G(Q,P)^t b(Q)\, d\Omega_Q \tag{2.11}$$

It is convenient to define as boundary conditions for the problem (2.4) the expression

$$(N^* + N')G(q,P) = 0 \tag{2.12}$$

so that the first integral in (2.11) vanishes. Adopting this procedure, a Green's function with Cauchy boundary conditions is defined for problem (2.1). The operator N' is defined as a diagonal matrix with constant elements k_i such that

$$k_i u_i(p) \cong 0 \text{ on } \partial\Omega_i \tag{2.13}$$

and k_i can be chosen as any non-zero value on the part of $\partial\Omega_i$ that has homogeneous Dirichlet boundary conditions.

The desired solution for $u(P)$ can then be written as

$$u(P) = \int_{\partial\Omega} G(q,P)^t F(q)\, d\partial\Omega_q + \int_{\Omega} G(Q,P)^t b(Q)\, d\Omega_Q \tag{2.14}$$

where $F(q) = (N + N')u(q)$.

Taking the trace theorem of $u(P)$, that is

$$u(p) = \lim_{P \to p} u(P) \quad p \in \partial\Omega, P \in \Omega, \tag{2.15}$$

in (2.14) leads to

$$u(p) = \int_{\partial\Omega} G(q,p)^t F(q)\, d\partial\Omega_q + \int_{\Omega} G(Q,p)^t b(Q)\, d\Omega_Q \tag{2.16}$$

The equations (2.14) and (2.16) define the problem completely and are the final integral expressions of the MLGFM to be discretized.

3. INTEGRAL EQUATIONS DISCRETIZATION

In equations (2.14) and (2.16) the domain variables are approximated by interpolating functions $[\Psi]$, in the same way as in the Finite Element Method. Similarly, the boundary variables are approximated by boundary interpolating functions $[\phi]$ like in the Boundary Element Method. Due to the application of the trace theorem, the interpolating functions for boundary displacements must be the trace of the domain ones.

Thus, the approximations can be summarized as

$$u(P) = [\Psi(P)]\{u\}^D \qquad F(q) = [\phi(q)]\{f\} \tag{3.1}$$

$$u(p) = [\phi(p)]\{u\}^C \qquad b(Q) = [\Psi(Q)]\{b\} \tag{3.2}$$

The next step is to substitute the expressions above into (2.14) and (2.16).

In the domain system obtained, the resultant residue is made orthogonal to each domain interpolation function, resulting in

$$A\{u\}^D = B\{f\} + C\{b\} \tag{3.3}$$

Similarly, in the boundary system obtained, the resultant residue is made orthogonal to each boundary interpolation function, leading to

$$D\{u\}^C = E\{f\} + F\{b\} \tag{3.4}$$

In equations (3.3) and (3.4) the following identities hold

$$A = \int_{\Omega}[\Psi(P)]^t[\Psi(P)]\, d\Omega_P \tag{3.5}$$

$$B = \int_{\partial\Omega}[Gd(q)]^t[\phi(q)]\, d\partial\Omega_q \tag{3.6}$$

$$C = \int_{\Omega}[Gd(Q)]^t[\Psi(Q)]\, d\Omega_Q \tag{3.7}$$

$$D = \int_{\partial\Omega}[\phi(p)]^t[\phi(p)]\, d\partial\Omega_p \tag{3.8}$$

$$E = \int_{\partial\Omega}[Gc(q)]^t[\phi(q)]\, d\partial\Omega_q \tag{3.9}$$

$$F = \int_{\Omega}[Gc(Q)]^t[\Psi(Q)]\, d\Omega_Q \tag{3.10}$$

where the Green's function projections involved are

$$[Gd(q)]^t = \int_{\Omega}[\Psi(P)]^t[G(q,P)]^t\, d\Omega_P \tag{3.11}$$

$$[Gd(Q)]^t = \int_{\Omega}[\Psi(P)]^t[G(Q,P)]^t\, d\Omega_P \tag{3.12}$$

$$[Gc(q)]^t = \int_{\partial\Omega}[\phi(p)]^t[G(q,p)]^t\, d\partial\Omega_p \tag{3.13}$$

$$[Gc(Q)]^t = \int_{\partial\Omega}[\phi(p)]^t[G(Q,p)]^t\, d\partial\Omega_p \tag{3.14}$$

4. APPROXIMATE GREEN'S FUNCTION PROJECTIONS

As it became evident in the last section, the MLGFM depends on the knowledge of four Green's function projections, given by (3.11) - (3.14). These Green's functions may be evaluated approximately, by solving two associate problems with the Finite Element Method as a residual procedure.

Problem 1:

$$A^*[Gd(Q)] = [\Psi(Q)] \tag{4.1}$$
$$(N^* + N')[Gd(q)] = 0 \quad \forall q \in \partial\Omega, Q \in \Omega \tag{4.2}$$

Problem 2:

$$A^*[Gc(Q)] = 0 \tag{4.3}$$
$$(N^* + N')[Gc(q)] = [\phi(q)] \quad \forall q \in \partial\Omega, Q \in \Omega \tag{4.4}$$

5. NUMERICAL RESULTS

Numerical comparisons were performed between the MLGFM and the FEM. For this purpose, quadratic finite elements of 20 and 27 nodes were employed in the domain discretization (Fig. 5.1). The boundary discretization in the MLGFM was done with quadratic boundary elements of 8 and 9 nodes (Fig. 5.2).

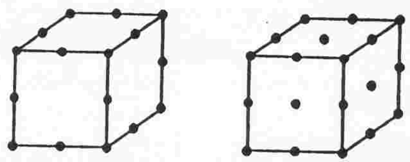

Fig. 5.1. Finite elements used for domain discretization.

Fig. 5.2. Boundary elements used for boundary discretization.

5.1. Bending of Curved Beam with a Load in its End

A curved beam (90 degrees) of thin rectangular transversal section is clamped in its face A and subjected to a radial load P applied on its face B, as can be seen in Fig. 5.3. The material of the beam is isotropic, its Poisson coefficient is $\nu = 0.2$ and the longitudinal elasticity modulus is $E = 1.0$.

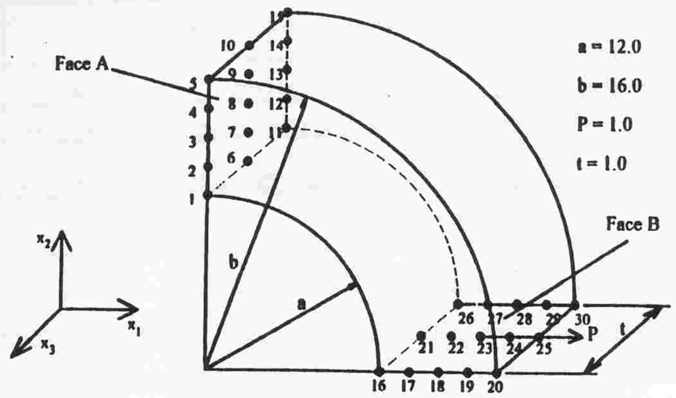

Fig. 5.3. Curved beam with applied load on face B

Two sets of domain and boundary meshes were used to analyse this problem. These are displayed in Fig. 5.4. Note that in this example, 27 node finite elements were used to represent the domain and 9 node boundary elements were used in the boundary. Two meshes were employed: **L10C34P2 - 10** Lagrangian domain elements, **34** Contour elements, Polinomial of order **2** and **L18C58P2 - 18** Lagrangian domain elements, **58** Contour elements, Polinomial of order **2**.

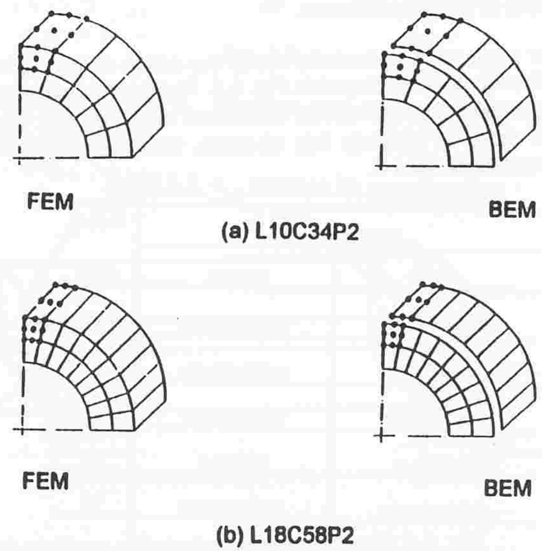

Fig. 5.4. Domain and boundary discretization.

The results for stresses and displacements are shown in Tables 5.1 and 5.2. The errors referenced to the 2-D solution are displayed in parenthesis.

Table 5.1. Radial displacement in the mean radius of the free extremity

L10C34P2		L18C58P2		2-D Analytical Solution (Timoshenko & Goodier, 1970)
MLGFM	FEM	MLGFM	FEM	
405.20778 (-0.91%)	405.19776 (-0.91%)	408.43083 (-0.12%)	408.41339 (-0.13%)	408.94028

Table 5.2. Normal stresses in the clamped face

r	L10C34P2		L18C58P2		2-D Analytical Solution (Timoshenko & Goodier, 1970)
	MLGFM	FEM	MLGFM	FEM	
16.0	-4.5984 (0.93%)	-5.1220 (12.42%)	-4.5536 (-0.05%)	-4.7447 (4.14%)	-4.5560
15.0	-2.5464 (5.58%)	-3.0096 (24.79%)	-2.4585 (1.94%)	-2.6135 (8.37%)	-2.4117
14.0	-0.0526 (-)	-0.5814 (-)	0.0001 (-)	-0.1753 (-)	0.0018
13.0	2.5769 (-7.37%)	2.1712 (-21.95%)	2.7239 (-2.08%)	2.5762 (-7.39%)	2.7819
12.0	6.1827 (1.78%)	5.5214 (-9.11%)	6.0901 (0.25%)	5.9267 (-2.44%)	6.0747

It may be observed from the Tables above that both MLGFM and FEM provide similar results for displacements. On the other hand, the normal stresses in the clamped face are much more accurately represented using the MLGFM.

5.2. Pure bending of a prismatic beam.

Consider a prismatic beam bent due to the action of two moments of equal magnitude and opposite directions as displayed in Fig. 5.5. The beam's material is considered isotropic, and the Young modulus and Poisson coefficient are given by $E = 2.10 \times 10^6$ and $\nu = 0.3$, respectively. In order to simulate this situation, face A is clamped and a bending stress distribution is applied on face B.

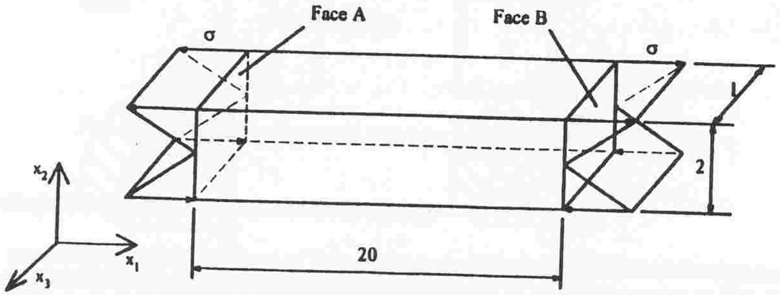

Fig. 5.5. Prismatic beam undergoing pure bending.

For this problem's modelling, 20 node finite elements were used in the domain and 8 node boundary elements were applied in the boundary. The mesh, presented in Fig. 5.6, is

called **S05C22P2** - **05** Serendipity domain elements, **22** Contour elements, Polinomial order **2**.

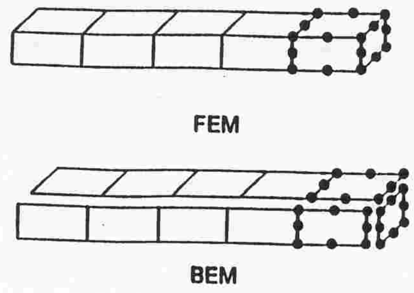

Fig. 5.6. Domain and boundary meshes for pure bending analysis (S05C22P2).

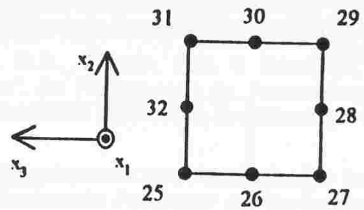

Fig. 5.7. Face A.

The results obtained for σ_{x2} stresses and maximum displacements are shown in Tables 5.3 and 5.4, respectively.

Table 5.3. σ_{x2} stresses for pure bending

Node	Mesh S05C22P2	
	MLGFM	FEM
25	-29,999.99997	-29,987.53498
26	-30,000.00001	-30,011.22807
27	-29,999.99999	-29,987.53497
28	0.000006	0.000021
29	29,999.99996	29,987.53498
30	30,000.00001	30,011.22807
31	29,999.99999	29,987.53497
32	-0.0000042714	-0.0000021225

Table 5.4. Maximum displacements for pure bending.

Displacement	Mesh S05C22P2 (Error %)		Analytical Solution (Timoshenko & Goodier, 1970)
	MLGFM	FEM	
u_{x_1}	-0.002142857 (7.467×10^{-9})	-0.000766180 (64.24)	-0.002142857
u_{x_2}	0.285714286 (3.5×10^{-7})	0.280515295 (1.82)	0.285714286
u_{x_3}	-2.856607143 (7.5×10^{-2})	-2.857014268 (1.7×10^{-1})	-2.858750000

In this example, the stress results are very similar for FEM and MLGFM. Conversely, the displacements are more accurate when calculated via MLGFM.

5.3. Bending of a uniformly loaded beam

Consider a beam with a rectangular cross section of unitary width which is simply supported in its ends and bent due to a uniform load of magnitude q=100, as displayed in Fig. 5.8. The beam's material is isotropic and the Young's Modulus and Poisson's coefficient are given respectively by $E = 2.1 \times 10^6$ and $\nu = 0.3$. The maximum displacement of the beam is presented in Table 5.5.

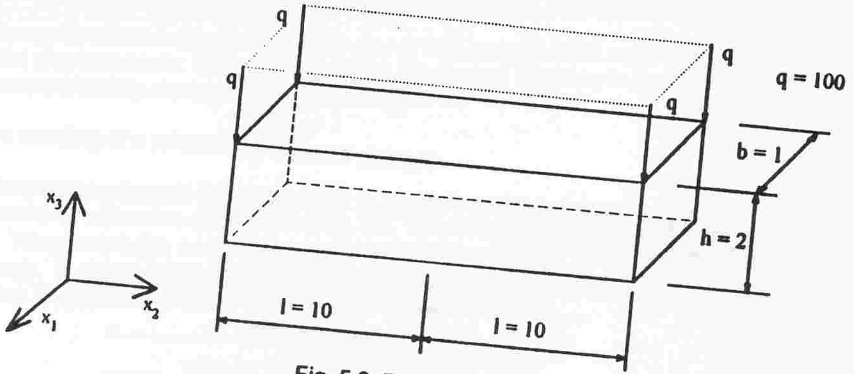

Fig. 5.8. Beam bending.

Table 5.5. Maximum displacement of the beam

L02C10P2		L10C42P2		2-D Analytical Solution (Timoshenko & Goodier, 1970)
MLGFM	FEM	MLGFM	FEM	
-0.13236 (-13.04%)	-0.13253 (-12.92%)	-0.15225 (0.03%)	-0.15242 (0.14%)	-0.15220

6. CONCLUSION

The numerical examples presented in this work show that the MLGFM is a powerful tool to obtain approximate solutions for 3-D elasticity problems. It can be observed that when compared to the FEM, the quality of the results given by the MLGFM may display a better behaviour. Example 5.1 agrees with the results presented by Muñoz R. & Barcellos (1994) and Barbieri et al. (1994), and shows that better results are normally obtained for tractions, while for displacements the results are usually the same. Nevertheless, in some ocasions one problem requires different boundary conditions when modelled with FEM and MLGFM. In this case, the property of better representation for tractions may be lost, as in example 5.2. Particularly, the better representation of stress results is foreseen to give a great advantage to the method for dealing with nonlinear problems, in which the quality of stresses plays a key role in the iterative process of solution.

7. REFERENCES

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