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GREEN'S FORMULATION FOR VISCOUS INCOMPRESSIBLE FLUID FLOW

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ABSTRACT

A Green's Function Method is used to obtain approximate solution to viscous incompressible fluid flow. Two auxiliary differential problems are locally defined in terms of specially defined Green's Functions associated to the problem. A Finite Element approximation is found for the projections of the Green's Function onto a finite dimensional subspace. This approximation is then used to construct a final algebraic system of equations, in terms of nodal values of the velocities, pressure and tractions on the boundary, which simulates the behaviour of the flow. Tests performed indicates an overall improvement on efficiency stresses approximations when compared to the Mixed FEM used.

1. INTRODUCTION

The accurate response prediction for the behaviour of viscous incompressible fluid flow is a fundamental step in the design of an immense variety of mechanical systems present in a chemical plant. It is observed that, in the case of analysis of solids, in general, the secondary variables, the stress components and tractions, are the main interest. In fluid flow analysis the interest mainly on the primary variables, the velocities and the pressure fields. Since most of the fluid flow problems require a nonlinear analysis, the accuracy of the approximations of the secondary variables plays a fundamental role, whether they are the final goal of the analyst or not. Whether the nonlinear iterations are defined by one of the Newton-like methods, by simple successive iterations or by other methods, the resultant linear approximate problem will consist of a coefficient matrix, and a right-hand side "force" vector. The components of these terms usually depend on several combinations of the displacements, stresses and its derivatives in each coordinate, evaluated at a known, previously computed configuration.

The use of more accurate approximations for the secondary variables results in fewer nonlinear iterations required to converge at a given load level. This represents an obvious improvement in numerical efficiency. In Finite Elements for solid analysis a great deal of effort has been made to improve or extract better values of stresses for a given finite element space, resulting in some more or less efficient techniques for displacement-based formulations, like the smoothing techniques, and the improvements on the basis functions, coupled with error estimates and automatic adaptivity of the mesh.

The first developments in the method were performed by Barcellos & Silva, 1987, Barbieri & Barcellos, 1991 and 1993. Since these early versions of the method, it has been applied successifully to a variety of problems like bending of plates and shells and the solution of the Helmholtz problem, among others. Mendonca, 1995, developed local forms for the method and introduced some procedures aimed to reduce the computational effort involved in the computations, specializing it to the efficient computation of secondary variables.

2. NAVIER-STOKES PROBLEM

As a prototype for incompressible problems, we consider the Navier-Stokes equations in the form

$$U\cdot\nabla U-\frac{\mu}{\rho}(\nabla^2U+(\nabla^2U)^t)+\frac{1}{\rho}\nabla p=\frac{1}{\rho}b \qquad (1)$$

on Ω, with the incompressible constraints and boundary conditions given by

$$\begin{aligned} \nabla\cdot U&=0 && \text{on } \Omega\\ U&=\bar{U} && \text{on } \Gamma_u \\ t&=\bar{t} && \text{on } \Gamma_f, \end{aligned} \qquad (2)$$

where p denotes the pressure, t the traction vector, Γ_u and Γ_f partitions of the boundary Γ such that $\Gamma=\Gamma_u\cup\Gamma_f$ and $\Gamma_u\cap\Gamma_f=\varnothing$. Also, $U=(u_1,u_2)$. As typically is done with the boundary element techniques, the nonlinearities will be handled mostly by the method of simple successive iterations. Therefore we alter Eqs.(1) to the form

$$-\mu(\nabla^2 U+(\nabla^2 U)^t)+\nabla p=b \qquad \text{on}\ \Omega, \tag{3}$$

where μ represent the inverse of the Reynolds number Re. In the remaining of this paper we will follow the main steps of the global Green's formalism described by Barbieri, 1991 and Mendonca 1995. For simplicity we will restrict the notation to the two-dimensional problem, but without implying any restriction to its extension to the full three dimensions.

First, for purpose of clarity, it is useful to express the system of equations (3) and (2a) in the operator form $Au=b$, where $u=(u_1,u_2,p)^t$, $b=(b_1,b_2,0)^t$ and

$$A=\begin{bmatrix} -\mu(\nabla\cdot\nabla(\cdot)+\partial_{11}(\cdot)) & -\mu\partial_{12}(\cdot) & \partial_1(\cdot) \\ -\mu\partial_{22}(\cdot) & -\mu(\nabla\cdot\nabla(\cdot)+\partial_{22}(\cdot)) & \partial_2(\cdot) \\ \partial_1(\cdot) & \partial_2(\cdot) & 0 \end{bmatrix}. \tag{4}$$

We premultiply it by the weighting vector function $v=(v_1,v_2,w)^t$, integrate both terms over the domain and perform an integration by parts on the first term obtaining

$$\frac{\mu}{2}\int_\Omega(\nabla V+(\nabla V)^t):(\nabla U+(\nabla U)^t)d\Omega-\oint_\Gamma v\cdot Nud\Gamma$$
$$-\int_\Omega p\nabla\cdot Vd\Omega=\int_\Omega V\cdot bd\Omega$$
$$\int_\Omega w\nabla\cdot Ud\Omega=0\ , \tag{5}$$

where $V=(v_1,v_2)$ and

$$N=\begin{bmatrix} \mu(2n_1\partial_1(\cdot)+n_2\partial_2(\cdot)) & \mu n_2\partial_1(\cdot) & -n_1 \\ \mu n_1\partial_2(\cdot) & \mu(2n_2\partial_2(\cdot)+n_1\partial_1(\cdot)) & -n_2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

This operator was formed reordering the definition of the tractions such that $t=S^t\cdot n=Nu$, where the stresses are given by the Navier-Poisson law of an incompressible Newtonian fluid: $S=-pI+\mu(\nabla U+(\nabla U)^t)$. Next we integrate by parts again the first term in both Equations (5) to get:

$$-\mu\int_\Omega U\cdot(\nabla\cdot\nabla V+\nabla\cdot(\nabla V)^t)d\Omega-\int_\Omega p\nabla\cdot Vd\Omega=$$
$$\oint_\Gamma v\cdot Nud\Gamma-\oint_\Gamma U\cdot N^*Vd\Gamma+\int_\Omega v\cdot bd\Omega$$
$$\int_\Omega u\cdot\nabla wd\Omega=-\oint_\Gamma wU\cdot nd\Gamma. \tag{7}$$

Now we add two terms to the boundary integrals through the arbitrary operator N' which satisfy the condition $v\cdot N'u-u\cdot N'v=0$. Here we choose the simplest form for this operator is $N'=Diag[\alpha(q),\beta(q),\gamma(q)]$. That this form satisfies the necessary condition is clear because $v\cdot N'u=\alpha v_1u_1+\beta v_2u_2+\gamma wp=u\cdot N'v$. Therefore Eqs.(7) can be simplified to the following operator form

$$\int_\Omega[u\cdot A^*v-v\cdot b]d\Omega=\oint_\Gamma[v\cdot(N+N')u-u\cdot(N^*+N')v]d\Gamma, \tag{8}$$

$$(N^*+N')=\begin{bmatrix} \mu(2n_1\partial_1(\cdot)+n_2\partial_2(\cdot))+\alpha & \mu n_2\partial_1(\cdot) & n_1 \\ \mu n_1\partial_2(\cdot) & \mu(2n_2\partial_2(\cdot)+n_1\partial_1(\cdot))+\beta & n_2 \\ 0 & 0 & \gamma \end{bmatrix} \tag{9}$$

and

$$A^*=\begin{bmatrix} -\mu(\nabla\cdot\nabla(\cdot)+\partial_{11}(\cdot)) & -\mu\partial_{12}(\cdot) & -\partial_1(\cdot) \\ -\mu\partial_{22}(\cdot) & -\mu(\nabla\cdot\nabla(\cdot)+\partial_{22}(\cdot)) & -\partial_2(\cdot) \\ -\partial_1(\cdot) & -\partial_2(\cdot) & 0 \end{bmatrix}. \tag{10}$$

Consider the third term in (8). The *apparent tractions* are defined as

$$F(q)=Nu(q)+N'u(q), \qquad q\in\Gamma. \tag{11}$$

Using the definitions of t and N' we can put it in the extended form:

$$\begin{bmatrix} F_1(q) \\ F_2(q) \\ F_3(q) \end{bmatrix}=\begin{bmatrix} t_1(q) \\ t_2(q) \\ 0 \end{bmatrix}+\begin{bmatrix} \alpha(q)u_1(q) \\ \beta(q)u_2(q) \\ \gamma(q)p(q) \end{bmatrix}. \tag{12}$$

As expected, in a boundary point of a two-dimensional domain only two physical components of tractions apply, those shown in the second bracket. The defined apparent tractions, however, contain a third component defined as shown in (12) in terms of the pressure and the arbitrary pre-defined functions $\gamma(q)$.

Once the problem is established in the form shown in (8) with the definitions shown in (12), the procedures described described by Mendonca, 1995, can be readily applied. Two auxiliary problems involving projections of the Green's Functions onto a finite dimensional subspace are defined by:

$$\begin{cases} A^*G_d(P)=[\psi(P)], & \forall P\in\Omega \\ (N^*+N')G_d(p)=0, & \forall p\in\Gamma \end{cases}$$
$$\begin{cases} A^*G_c(P)=0, & \forall P\in\Omega \\ (N^*+N')G_c(p)=[\psi(p)], & \forall p\in\Gamma \end{cases}. \tag{13}$$

The domain and the boundary are partitioned into a set of elements. The fields are approximated by $u(Q)=[\psi(Q)]u^d$, with

$$[\psi(Q)]=Diag[\delta_1(Q),\delta_2(Q),\delta_3(Q)], \tag{14}$$

where each $\delta_i(Q), i=1,2,3$, is a set of nodal basis functions and u^d is the column vector formed by the nodal values, i.e., $u^d=(u_1^d,u_2^d,u_3^d)^t$. Replacing the index d by c in Eqn. (14), analogous expressions are defined for the fields on the boundary, using suitable basis functions as given by Mendonca, 1995.

For two vector fields defined by $z=(Z,z_3)^t$ and $v=(V,v_3)^t$ with $Z=(z_1,z_2)^t$ and $V=(v_1,v_2)^t$, the bilinear form shown in the associated weak form associated with the first Auxiliary problem shown in Eqn. (13) is defined by

$$B(z,v)=\frac{\mu}{2}\int_{\Omega}\left(\nabla Z+(\nabla Z)^{t}\right):\left(\nabla V+(\nabla V)^{t}\right)d\Omega-\int_{\Omega}z_{3}\nabla\cdot V d\Omega+\int_{\Omega}v_{3}\nabla\cdot Z d\Omega. \quad (15)$$

The associated coefficient matrix K_o can be obtained in the form usual in the finite element method. Its partitioned form, compatible to the ordering of the degrees of freedon used in Eqn. (14) is:

$$K_o=\begin{bmatrix} k^{11} & k^{12} & k^{13} \\ k^{21} & k^{22} & k^{23} \\ k^{31} & k^{32} & 0 \end{bmatrix}, \quad (16)$$

The boundary condition in the problem (13) will result in another coefficient matrix called K_2. The addition of K_2 to K_o provides sufficient restrictions to allow for the solution of the nodal values of the Green's Functions Projections, G^{DP} and G^{CP} by solving the following algebraic system derived from (13):

$$[K_o+K_2][G^{DP};G^{CP}]=[M;m]. \quad (17)$$

The projections allow the definition of a Modified Somigliana's Identity after Eqn. (12), which can be discretized on the boundary to result in the final algebraic systems of equations for the problem, in terms of the nodal values of the velocities, pressure and tractions on the boundary:

$$Du^{c}=EF^{c}+Fb^{d} \quad (18)$$

where, similarly to the boundary element method, u^{c} is the set of nodal values of velocities and pressure at the boundary, F^{c} is the set of nodal values of the tractions defined in Eqn. (16), aand the matrices are given in terms of the nodal values of the Green's Function Projections by (21a):

$$D=\oint_{\Gamma_q}[\phi(q)]^{t}[\phi(q)]d\Gamma_q;\ E=\oint_{\Gamma_p}G_c(p)^{t}[\phi(p)]d\Gamma_p;$$

$$F=\int_{\Omega_p}G_c(P)^{t}[\psi(P)]d\Omega_P \quad (19)$$

$$G_c(p)^{t}=\oint_{\Gamma_p}[\phi(p)]^{t}G_c(p,q)^{t}d\Gamma_p;G_c(P)^{t}=\oint_{\Gamma_p}[\phi(p)]^{t}G_c(P,q)^{t}d\Gamma_p$$

where $G_c(p)$ and $G_c(P)$ are the projections of the Green's Functions onto finite dimensional spaces, solutions to the Auxiliary Problems (13). All the derivations and expressions shown so far can be applied to the whole physical domain, but here we will be using then in the sense described in Mendonca, 1995, i.e., each cell is modelled individualy, its local projections are computed from (17) and the matrices in (18) are assembled to form the algebraic systems for the complete domain.

3 - SPURIOUS PRESSURE MODES

In this section we describe some aspects that, although unsatisfactory due to its incompleteness, seems interesting enough to deserve a record in view of possible future developments. We notice that the addition ot K_2 to K_o make the submatrix (3,3) appearing in Eqn. (17) partially populated when the arbitrary function $\gamma(q)$ is not defined to be identically zero at all points of the boundary. In case the projections are being computed at a cell composed by a single element, and if the basis functions are defined in this element by nodes distributed only on its boundary, like the elements of the Serendipty family, then the submatrix (3,3) in Eqs.(17) will be fully populated by a positive-definite matrix (if $\gamma(q)>0$ for any $q\in\Gamma$).

It is observed that the approximate response obtained is, within the easily met restrictions, completely independent of the particular values chosen for the arbitrary functions, in particular $\gamma(q)$, in the definitions of k_{33}.

As a consequence, the algebraic system (18) will be solvable even in some of the cases when the underlying finite element used presents spurious pressure modes. Some finite element formulation with *forbidden* combinations of basis functions for velocities and pressure were chosen and tested to identify models where they would fail. A second effect, however, renders the above observations temporarily useless: even if the problem (17) was solvable for the projections, the procedure would fail in the next step. When using the div-unstable underlying finite elements to compute K_o, the obtained set of projections has less than the required rank. The consequence is that the final algebraic system, Eqn. (18), will generate a singular final matrix after the boundary conditions have been applied.

4 - APPLICATIONS

The numerical tests were performed with the Green's macrocells composed of one single domain element with biquadratic basis functions for velocities and bilinear functions for pressure. Three sets of problems were solved. In the first two the standard Couette and Poiseuille flows were modeled. Because the exact solutions for these problems are bilinear and biquadratic polynomials for velocities and bilinear polynomials for pressure respectively, the approximate responses obtained were exact as expected, in all fields of velocities, pressure and stresses.

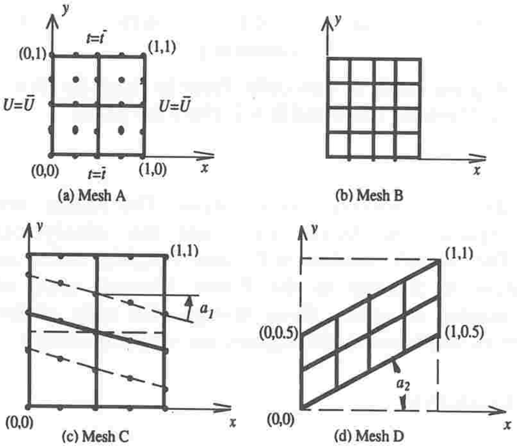

Figura 1 - Malhas utilizadas.

Next, we took one of the few problems for which an analytical solution is available, concentric rotating cylinders (see Schlichting), and truncated a standard 1x1 domain to be modeled. The analytical solution is:

$$u_1(x,y) = -U_1 k[r_2 - \frac{R^2}{r_2}]\frac{y}{R^2}, u_2(x,y) = -U_1 k[r_2 - \frac{R^2}{r_2}]\frac{y}{R^2}$$

$$p(x,y) = \frac{\rho(U_1 k)^2}{2}[\left(\frac{R}{r_2}\right)^2 - 4\log(\frac{R}{r_2}) - \left(\frac{r_2}{R}\right)^2] + C, \quad (20)$$

where $r_1 = 1/3$, $r_1 = 5/3$, $U_1 = 1$, $p_1 = 0.0$, $\mathrm{Re} \equiv \frac{\rho U_1 L}{\mu} = 0.1$ $d = r_1 / r_2$, $k = d/(1-d^2)$ and $R^2 = (x+r_1)^2 + y^2$. The standard domain Ω considered and the position of the frame of reference is shown in Figure 1. The particular choice of parameters will render $u_1 = 0$ along the face $y = 0$, $u_2(0,0) = U_1$ and $u_1(1,1) = u_2(1,1) = 0$. The meshes used are displayed in Figure 1. Figures 2 to 4 show the stress components S_x and S_{xy} along the lines $x = 0$ and $x = 0.5$. The quantification of the errors is made through the use of a modified L^2-norm of error, defined along a given segment of line. The relative error E for S_x is given by

$$E = \frac{\int_{y=0}^{1} [S_x(0,y) - S_{xo}(0,y)]^2 \, dy}{\int_{y=0}^{1} S_{xo}(0,y)^2 \, dy}, \quad (21)$$

where S_{xo} is the analytic solution obtained from (21). The relative error for S_x is defined analogously.

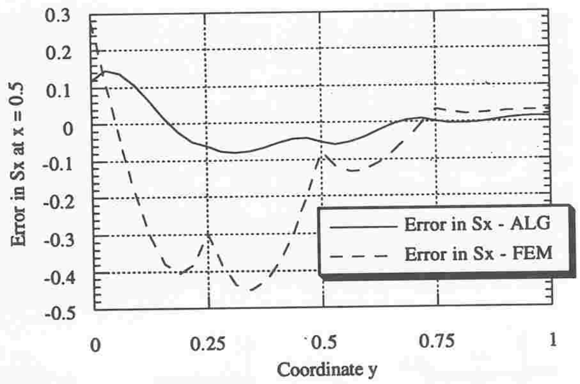

Figure 2 - Regurar mesh B, 4x4 cells. Error in stress *Sx* at *x* = *0.5.*. E = 1.37e-4 for ALG and E = 2.19e-3 for FEM.

ALG stands for *Assembled Local Green*. The results are compared against the Mixed FEM and the alnalytical solution. The Green's meshes will have roughly twice as many degrees of freedom as the Finite Element mesh of identical number of nodes. Even though, the ratio of the relative errors displayed in the figures are very expressive.

5. CONCLUSIONS

The application of the Green's Function Method has been briefly outlined. This procedure lead to : reduced computational effort; for meshes of equal sizes it approximates the stresses more accurately and presents higher rate of convergence than the mixed FEM, suggesting further developments towards the use in continuum mechanics problems

6. REFERENCES

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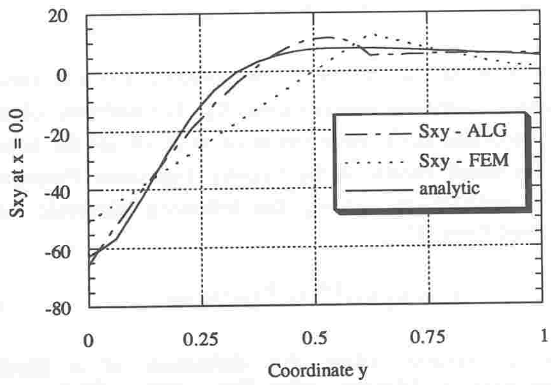

Figure 3 - Stress *Sxy* at *x* = *0.0.* for 2x2 distorted mesh C. E = 1.44e-2 for ALG and E = 1.55e-1 for FEM.

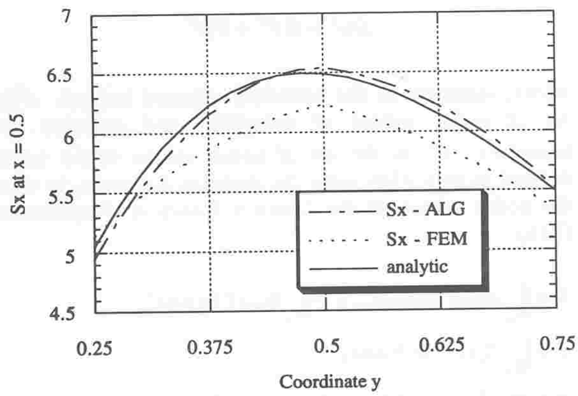

Figure 4 - Stress *Sx* at *x* = *0.5.* for 4x2 distorted mesh D. E = 1.35e-4 for ALG and E = 2.02e-3 for FEM.

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